## 5.1 Second-Order linear PDE

Consider a second-order linear PDE

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \qquad (x,y) \in U \tag{5.1}$$

for an unknown function $u$ of two variables $x$ and $y$. The functions $a, b$ and $c$ are assumed to be of class $C^1$ and satisfying $a^2+b^2+c^2 \neq 0$. The operator

$$L_0[u] := au_{xx} + 2bu_{xy} + cu_{yy}$$

consisting of the second order terms of $L$ is called the *principal part* of $L$. Many of the fundamental properties of the solutions of (5.1) are determined by the sign of the discriminant of $L$. The *discriminant* $\Delta(L)(x,y)$ is defined by

$$\Delta(L)(x,y) = \det \begin{bmatrix} b & a \\ c & b \end{bmatrix} = b^2 - ac$$

where $a, b$ and $c$ are evaluated at the point $(x,y)$.

We are interested in how the PDE is transformed under changes of coordinates. We consider a $C^1$- map $F(x,y) = (\xi(x,y), \eta(x,y))$ whose Jacobian satisfies

$$\det J(x,y) = \det \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \neq 0$$

at each point of $(x_0,y_0) \in U$. The inverse function theorem implies that near the point $(x_0,y_0)$ the map $F$ has an inverse $F^{-1}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta))$. The inverse is of class $C^1$. Now, assuming that $u$ is a solution of (5.1), define $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$. Then $u(x,y) = w(\xi(x,y), \eta(x,y))$ and, by the chain rule,

$$\begin{aligned}
u_x &= w_\xi \xi_x + w_\eta \eta_x \\
u_y &= w_\xi \xi_y + w_\eta \eta_y \\
u_{xx} &= w_{\xi\xi}(\xi_x)^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}(\eta_x)^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx} \\
u_{yy} &= w_{\xi\xi}(\xi_y)^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}(\eta_y)^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy} \\
u_{xy} = u_{yx} &= w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + w_{\eta\eta}\xi_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}
\end{aligned}$$

Substituting into (5.1), we find that

$$\widetilde{L}[w] = Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G \tag{5.2}$$

with the coefficients of the principal part $\widetilde{L}_0[w] = Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta}$ given by

$$\begin{aligned} A(\xi,\eta) &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\ B(\xi,\eta) &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \\ C(\xi,\eta) &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \end{aligned} \tag{5.3}$$

Observe that

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_y & \eta_y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \cdot \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix}^t,$$

where $^t$ denotes the transpose of the matrix. Recalling that the determinant of the product of matrices is equal to the product of the determinants of matrices and that the determinant of a transpose of a matrix is equal to the determinant of a matrix, we get

$$\det \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det \begin{bmatrix} a & b \\ b & c \end{bmatrix} \cdot (J(x,y))^2. \tag{5.4}$$

This shows that the discriminant of $L$ has the same sign as the discriminant of the transformed equation and so it is an invariant of the change of coordinates. Consequently, we can classify equations (5.1) according to the sign of the discriminant.

**Definition 5.1.** The equation (5.1) is called

- *hyperbolic* at $(x,y)$ if $\Delta(L)(x,y) > 0$.
- *parabolic* at $(x,y)$ if $\Delta(L)(x,y) = 0$.
- *elliptic* at $(x,y)$ if $\Delta(L)(x,y) < 0$.

**Example 5.2.** (i) The wave equation $u_{tt} - u_{xx} = 0$ is a hyperbolic equation.

(ii) The heat equation $u_t - u_{xx} = 0$ is a parabolic equation.

(iii) The Laplace equation $u_{xx} = u_{yy} = 0$ is an elliptic equation.

**Example 5.3.** Consider the Tricomi equation

$$yu_{xx} + u_{yy} = 0. \tag{5.5}$$

Here $a = y, b = 0, c = 1$ and $d = e = f = g = 0$. Its discriminant is equal to

$$\det \begin{bmatrix} b & a \\ c & b \end{bmatrix} = \det \begin{bmatrix} 0 & y \\ 1 & 0 \end{bmatrix} = -y.$$

Hence the equation (5.5) is hyperbolic for $y < 0$, parabolic when $y = 0$, and elliptic for $y > 0$.

Next we shall show that we can find changes of coordinates in which the (5.1) takes a simple form.

- **Hyperbolic equations**.

Suppose that equation (5.1) is hyperbolic on the domain $U$. This means that $b^2 - ac > 0$ at each point of $U$. We shall show that in this we can choose $(x, y) \mapsto (\xi(x, y), \eta(x, y))$ so that

$$A(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \tag{5.6}$$

$$C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0. \tag{5.7}$$

Under such a change of coordinates and dividing by $2B$ the hyperbolic equation (5.1) takes its *canonical form*

$$\widetilde{L}[w] = w_{\xi\eta} + \ell[w] = G, \tag{5.8}$$

where $\ell$ is a first-order linear operator and $G$ is function.
Note that if $a$ and $c$ are equal to 0, then the equation (5.1) is already in its canonical form (just divide by $2b$). Hence without loss of generality we may assume that $a \neq 0$. Note also that the equation (5.7) for $\eta$ is the same as (5.6) for $\xi$. Hence it suffices to consider only one of the equations, say (5.6). It can be written as a product

$$a\left[\xi_x - \frac{-b - \sqrt{b^2 - ac}}{a}\xi_y\right] \cdot \left[\xi_x - \frac{-b + \sqrt{b^2 - ac}}{a}\xi_y\right] = 0$$

and so, we need to solve the following two linear equations

$$\xi_x - \mu_1\xi_y = 0 \tag{5.9}$$

and

$$\xi_x - \mu_2\xi_y = 0, \tag{5.10}$$

where we have abbreviated

$$\mu_1 = \frac{-b - \sqrt{b^2 - ac}}{a} \quad \text{and} \quad \mu_2 = \frac{-b - \sqrt{b^2 - ac}}{a}.$$

Note that $\mu_1$ and $\mu_2$ are the real solutions of the equation

$$a\mu^2 + 2b\mu + c = 0. \tag{5.11}$$

In order to obtain a nonsingular map $(x,y) \mapsto (\xi(x,y), \eta(x,y))$, we choose $\xi$ to be the solution of (5.9) and $\eta$ to be the solution of (5.10). To solve (5.9), we use the method of characteristics (except that we don't specify the initial condition). The characteristic equations are

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = -\mu_1, \qquad \frac{dz}{dt} = 0.$$

The last equation says that the solution $\xi$ is constant along each of the characteristics $(x(t), y(t))$. In view of the first two equations, the characteristics can be obtain as curves $y = y(x)$ solving

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\mu_1, \tag{5.12}$$

and then the solution $\xi$ is constant at points $(x, y(x))$. Similarly, one solves (5.10) to obtain $\eta$.

In summary, to choose $\xi$ and $\eta$ one solves the (5.11) to obtain two real roots $\mu_1$ and $\mu_2$. Then, denoting by

$$f(x,y) = C_1 \qquad \text{and} \qquad g(x,y) = C_2$$

the solutions of characteristics equations

$$\frac{dy}{dx} = -\mu_1(x,y \quad \text{and} \qquad \frac{dy}{dx} = -\mu_2(x,y), \tag{5.13}$$

the variables $\xi$ and $\eta$ are defined by

$$\xi(x,y) = f(x,y), \qquad \eta(x,y) = g(x,y).$$

The solutions of both equations in (5.13) are called the two families of *characteristics* of (5.1).

**Example 5.4.** Consider

$$yu_{xx} + u_{yy} = 0$$

In the region where $y < 0$, the equation is hyperbolic. Solving $y\mu^2 + 1 = 0$, one finds two real solutions

$$\mu_1 = -\frac{1}{(-y)^{1/2}} \quad \text{and} \quad \mu_2 = \frac{1}{(-y)^{1/2}}$$

We look for two real families of characteristics, $\frac{dy}{dx} + \mu_1 = 0$ and $\frac{dy}{dx} + \mu_2 = 0$,

$$\frac{dy}{dx} - \frac{1}{(-y)^{1/2}} = 0 \quad \text{and} \quad \frac{dy}{dx} + \frac{1}{(-y)^{1/2}} = 0.$$

The solutions of the equations are

$$\frac{2}{3}(-y)^{3/2} + x = C_1 \qquad \text{and} \qquad -\frac{2}{3}(-y)^{3/2} + x = C_2.$$

Therefore, we set

$$\xi = \frac{2}{3}(-y)^{3/2} + x \quad \text{and} \quad \eta = -\frac{2}{3}(-y)^{3/2} + x.$$

The derivatives of $\xi$ and $\eta$ are,

$$\xi_x = 1, \quad \xi_y = -(-y)^{1/2}, \quad \eta_x = 1, \quad \eta_y = (-y)^{1/2}.$$

With $u(x, y) = v(\xi(x, y), \eta(x, y)$, one gets

$$\begin{aligned}
u_x &= v_\xi + v_\eta \\
u_y &= -(-y)^{1/2} v_\xi + (-y)^{1/2} v_\eta \\
u_{xx} &= v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta} \\
u_{yy} &= -y v_{\xi\xi} + 2y v_{\xi\eta} - y v_{\eta\eta} + \frac{1}{2}(-y)^{-1/2}[v_\xi - v_\eta].
\end{aligned}$$

Substituting into the equation, one obtains

$$\begin{aligned}
0 &= y u_{xx} + u_{yy} \\
&= 4y v_{\xi\eta} + \frac{1}{2}(-y)^{-1/2}[v_\xi - v_\eta] = 4y \left[ v_{\xi\eta} - \frac{1}{8}(-y)^{-3/2}(v_\xi - v_\eta) \right].
\end{aligned}$$

Since $\xi - \eta = \frac{4}{3}(-y)^{3/2}$, one concludes that

$$v_{\xi\eta} - \frac{1}{6(\xi - \eta)}(v_\xi - v_\eta) = 0.$$

- **Parabolic equations**.

Suppose that (5.1) is parabolic on the domain $U$. Hence $b^2 - ac = 0$ at each point of $U$. As before assume that $a \neq 0$ on $U$. We find a map $(x, y) \mapsto (\xi x, y), \eta(x, y))$ so that $B(\xi, \eta) = A(\xi, \eta) = 0$. It suffices to make $A = 0$ since $0 = B^2 - AC = B^2$ implies that $B(\xi, \eta) = 0$. Under such a change of coordinates the parabolic equation (5.1) can be brought to its *canonical form*

$$\widetilde{L}[w] = w_{\xi\xi} + \ell[w] = G(\xi, \eta)$$

where $\ell$ is a first-order linear operator and $G$ is function.

To do this we look for $(x, y) \mapsto \xi(x, y)$ so that

$$A(\xi, \eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0.$$

Since $b^2 = ac$, we have

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\left(\xi_x^2 + 2\frac{b}{a}\xi_x\xi_y + \frac{b^2}{a^2}\xi_y^2\right) = a\left(\xi_x - \mu\xi_y\right)^2,$$

where $\mu = -\frac{b}{a}$ is the double root of

$$a\mu^2 + 2b\mu + c = 0.$$

So, we look for the solution $\xi$ of the first-order linear equation,

$$\xi_x - \mu\xi_y = 0.$$

The solution $\xi$ is constant along each characteristic which is determined by the equation

$$\frac{dy}{dx} = -\mu = \frac{b}{a}. \tag{5.14}$$

For the map $(x, y) \mapsto \eta(x, y)$ we can take any map so that $\xi_x\xi - \xi_y\xi_x \neq 0$.

In summary, in the parabolic case to choose $\xi$ and $\eta$ one solves the (5.11) to obtain double root $\mu = -b/a$. Then, denoting by

$$f(x, y) = C$$

the solution of characteristics equation

$$\frac{dy}{dx} = -\mu \tag{5.15}$$

the variable $\xi$ is defined by

$$\xi(x, y) = f(x, y)$$

and the variable $\eta$ is chosen so that $\xi_x\xi - \xi_y\xi_x \neq 0$.

**Example 5.5.** *Reduce the following equation to its canonical form and then find the general solution,*

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$$

*for* $x > 0$.

The discriminant is equal to

$$\det \begin{bmatrix} -xy & x^2 \\ y^2 & -xy \end{bmatrix} = 0$$

so that the equation is parabolic. The quadratic equation $x^2\mu^2 - 2xy\mu + y^2 = 0$ has exactly one solution, $\mu = \frac{y}{x}$. Next we look for characteristics. These are solutions of the equation

$$\frac{dy}{dx} = \mu, \quad \text{i.e.,} \ \frac{dy}{dx} = -\frac{y}{x}.$$

The family of solutions is given by $xy = C$. Therefore, we define $\xi(x, y) = xy$ and we take as the second independent variable $\eta(x, y) = x$. Then

$$\xi_x = y, \quad \xi_y = x, \quad \eta_x = 1, \quad \eta_y = 0.$$

The Jacobian of the map $(x, y) \mapsto (\xi(x, y), \eta(x, y))$ is nonzero. Let $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, that is, $u(x, y) = v(\xi(x, y), \eta(x, y))$. Using the chain rule,

$$\begin{aligned} u_x &= yv_\xi + v_\eta \\ u_y &= xv_\xi \\ u_{xx} &= y^2 v_{\xi\xi} + 2yv_{\xi\eta} + v_{\eta\eta} \\ u_{xy} &= xyv_{\xi\xi} + xv_{\xi\eta} + v_\xi \\ u_{yy} &= x^2 v_{\xi\xi}. \end{aligned}$$

Substituting into the equation, one gets

$$x^2 v_{\eta\eta} + xv_\eta = 0,$$

so that, using $x = \eta$,

$$v_{\eta\eta} + \frac{1}{\eta} v_\eta = 0.$$

To solve this equation introduce the function $w = v_\eta$. Then

$$w_\eta = -\frac{1}{\eta} w$$

which has the general solution $w = \frac{1}{\eta} A(\xi)$. So,

$$v_\eta = \frac{1}{\eta} A(\xi)$$

which after integration gives

$$v(\xi, \eta) = A(\xi) \ln \eta + B(\xi).$$

Since $\xi(x, y) = xy$ and $\eta(x, y) = x$ and $u(x, y) = v(\xi(x, y), \eta(x, y))$, the general solution of the equation has the form

$$u(x, y) = A(xy) \ln x + B(xy)$$

where $A, B$ are two arbitrary functions of class $C^2$.

- **Elliptic equations**.

Suppose that (5.1) is elliptic on the domain $U$. Then $b^2 - ac < 0$ at each point of $U$. This time we look for the map $(x, y) \mapsto (x\xi(x, y), \eta(x, y))$ so that

$$\begin{aligned} A(\xi, \eta) &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = C(\xi, \eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\ B(\xi, \eta) &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0. \end{aligned}$$

Under such a change of coordinates and dividing by $A$, the elliptic equation (5.1) can be brought to its *canonical form*

$$\widetilde{L}[w] = w_{\xi\xi} + w_{\eta\eta} + \ell[w] = G(\xi, \eta)$$

where $\ell$ is a first-order linear operator.

The above system consists of two nonlinear first-order equations. Subtracting $C(\xi, \eta)$ form $A(\xi, \eta)$ and multiplying $B(\xi, \eta)$ by $2i$ leads to the following system,

$$\begin{gathered} a(\xi_x^2 - \eta_x^2) + 2b(\xi_x\xi_y - \eta_x\eta_y) + c(\xi_y^2 - \eta_y^2) = 0 \\ a\xi_x(2i\eta_x) + b(\xi_x(2i\eta_y) + \xi_y(2i\eta_x)) + c\xi_y(2i\eta_y) = 0. \end{gathered}$$

Consequently, setting $\phi = \xi + i\eta$, one finds that the above system is equivalent to

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0.$$

This can be written as a product,

$$a\,[\phi_x - \mu_1\phi_y] \cdot [\phi_x - \mu_2\phi_y] = 0,$$

where we have abbreviated

$$\mu_1 = \frac{-b - i\sqrt{ac - b^2}}{a} \qquad \text{and} \qquad \mu_2 = \frac{-b - i\sqrt{ac - b^2}}{a}.$$

These are complex roots of

$$a\mu^2 + 2b\mu + c = 0.$$

Note that $\mu_1$ and $\mu_2$ are conjugated, i.e., $\mu_1 = \overline{\mu_2}$. As in the hyperbolic case we solve the characteristics equations

$$\frac{dy}{dx} = -\mu_1 \qquad \text{and} \qquad \frac{dy}{dx} = -\mu_2. \tag{5.16}$$

This time the solutions are complex. If $f(x, y) = C_1$ and $g(x, y) = C_2$ are complex solutions of (5.16), then

$$\phi(x, y) = f(x, y) \quad \text{and} \quad \psi(x, y) = g(x, y).$$

Then set

$$\xi = \frac{1}{2}(\phi + \psi) \quad \text{and} \quad \eta = \frac{1}{2i}(\phi - \psi).$$

**Example 5.6.** Consider

$$yu_{xx} + u_{yy} = 0$$

In the region where $y > 0$, the equation is elliptic. Solving $y\mu^2 + 1 = 0$, one finds two complex solutions

$$\mu_1 = \frac{i}{y^{1/2}} \quad \text{and} \quad \mu_2 = -\frac{i}{y^{1/2}}$$

We look for two complex families of characteristics, $\frac{dy}{dx} + \mu_1 = 0$ and $\frac{dy}{dx} + \mu_2 = 0$,

$$\frac{dy}{dx} + \frac{i}{y^{1/2}} = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{i}{y^{1/2}} = 0.$$

The solutions of the equations are

$$\frac{2}{3}y^{3/2} + ix = C_1 \qquad \text{and} \qquad \frac{2}{3}y^{3/2} - ix = C_2.$$

Therefore, we set

$$\phi = \frac{2}{3}y^{3/2} + x \qquad \text{and} \qquad \psi = \frac{2}{3}y^{3/2} - x,$$

and then

$$\xi = \frac{1}{2}(\phi + \psi) = \frac{2}{3}y^{3/2} \qquad \text{and} \qquad \eta = \frac{1}{2i}(\phi - \psi) = x.$$

The derivatives of $\xi$ and $\eta$ are,

$$\xi_x = 0, \qquad \xi_y = y^{1/2}, \qquad \eta_x = 1, \qquad \eta_y = 0.$$

With $u(x,y) = v(\xi(x,y), \eta(x,y)$, one gets

$$\begin{aligned} u_x &= v_\eta \\ u_y &= y^{1/2} v_\xi \\ u_{xx} &= v_{\eta\eta} \\ u_{yy} &= y v_{\xi\xi} + \frac{1}{2} y^{-1/2} v_\xi. \end{aligned}$$

Substituting into the equation, one obtains

$$v_{\xi\xi} + v_{\eta\eta} + \frac{1}{2y^{3/2}} v_\xi = 0.$$

Finally, since $\xi = \frac{2}{3} y^{3/2}$, the equation becomes

$$v_{\xi\xi} + v_{\eta\eta} + \frac{3}{\xi} v_\xi = 0.$$