SINGULAR CARDINALS AND THE PCF THEORY

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1. Introduction.

Among the most remarkable discoveries in set theory in the last quarter century is the rich structure of the arithmetic of singular cardinals, and its deep relationship to large cardinals. The problem of finding a complete set of rules describing the behavior of the continuum function $2^{\aleph_\alpha}$ for singular $\aleph_\alpha$'s, known as the *Singular Cardinals Problem*, has been attacked by many different techniques, involving forcing, large cardinals, inner models, and various combinatorial methods. The work on the singular cardinals problem has led to many often surprising results, culminating in a beautiful theory of Saharon Shelah called the pcf theory ("pcf" stands for "possible cofinalities"). The most striking result to date states that if $2^{\aleph_n} < \aleph_\omega$ for every $n = 0, 1, 2, \ldots$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

In this paper we present a brief history of the singular cardinals problem, the present knowledge, and an introduction into Shelah's pcf theory. In Section 2, 3 and 4 we introduce the reader to cardinal arithmetic and to the singular cardinals problems. Section 5, 6, 7 and 8 describe the main results and methods of the last 25 years and explain the role of large cardinals in the singular cardinals problem. In Section 9 we present an outline of the pcf theory.

2. The arithmetic of cardinal numbers.

Cardinal numbers were introduced by Cantor in late 19^{th} century and problems arising from investigations of rules of arithmetic of cardinal numbers led to the birth of set theory.

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The operations of addition, multiplication and exponentiation of infinite cardinal numbers are a natural generalization of such operations on integers. Addition and multiplication of infinite cardinals turns out to be simple: when at least one of the numbers κ, λ is infinite then both $\kappa+\lambda$ and $\kappa\cdot\lambda$ are equal to $\max\{\kappa, \lambda\}$. In contrast with $+$ and $\cdot$, exponentiation presents fundamental problems. In the simplest nontrivial case, 2^κ represents the cardinal number of the power set $P(\kappa)$, the set of all subsets of κ. (Here we adopt the usual convention of set theory that the number κ is identified with a set of cardinality κ, namely the set of all ordinal numbers smaller than κ. In this representation, the cardinal number $\aleph_\alpha$ is the same as the ordinal number ω_α.) By a celebrated theorem of Cantor, $2^\kappa > \kappa$ holds for all cardinals κ, and therefore $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for every infinite cardinal $\aleph_\alpha$. In [6], Cantor conjectured that $2^{\aleph_0} = \aleph_1$, which became known as the *Continuum Hypothesis* (and the similar conjecture $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ as the Generalized Continuum Hypothesis or GCH).

It has soon become apparent that if GCH were true, one could completely describe rules for all of cardinal arithmetic, including infinite sums and products ($\sum_{i\in I}\kappa_i$ and $\prod_{i\in I}\kappa_i$) of cardinal numbers. Despite efforts of Cantor himself and others, the question whether GCH, or even CH, is true, remained unanswered until the emergence of methods of modern logic.

For a long time, the only source of inequalities in cardinal arithmetic was König's Theorem [36]. The theorem states that if $\{\kappa_i : i \in I\}$ and $\{\lambda_i : i \in I\}$ are two indexed families of cardinal numbers such that $\kappa_i < \lambda_i$ for all i, then

$$\sum_{i\in I}\kappa_i < \prod_{i\in I}\lambda_i\ .$$

Note that Cantor's Theorem itself is a special case of this: letting $\kappa_i = 1$ and $\lambda_i = 2$ for all $i < \kappa$, we get $\kappa = \sum_{i<\kappa} 1 < \prod_{i<\kappa} 2 = 2^\kappa$.

König's Theorem also provides important information about singular cardinals. Let us recall that the *cofinality* of an infinite cardinal $\aleph_\alpha$ is the least cardinal $\aleph_\gamma = \mathrm{cf}\,\aleph_\alpha$ such that $\aleph_\alpha$ is the supremum of an increasing $\aleph_\gamma$-sequence of ordinal numbers. A cardinal $\aleph_\alpha$

is *regular* if $\mathrm{cf}\,\aleph_\alpha = \aleph_\alpha$ and *singular* if $\mathrm{cf}\,\aleph_\alpha < \aleph_\alpha$. As $\mathrm{cf}(\mathrm{cf}\,\aleph_\alpha) = \mathrm{cf}\,\aleph_\alpha$, the cofinality is always a regular cardinal. Each successor cardinal $\aleph_{\alpha+1}$ is regular, as is the countable cardinal $\aleph_0$.

One consequence of König's Theorem is that for every α and every β, $\mathrm{cf}\,\aleph_\alpha^{\aleph_\beta} > \aleph_\beta$. It follows easily that when $\aleph_\beta \geq \mathrm{cf}\,\aleph_\alpha$ then $\aleph_\alpha^{\aleph_\beta} \neq \aleph_\alpha$, and therefore $\aleph_\alpha^{\aleph_\beta} > \aleph_\alpha$. Using this as well as rather elementary rules for cardinal exponentiation due to Hausdorff and Tarski, one obtains a complete set of rules for exponentiation under the assumption of GCH:

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \text{if } \aleph_\beta < \mathrm{cf}\,\aleph_\alpha \\ \aleph_{\alpha+1} & \text{if } \mathrm{cf}\,\aleph_\alpha \leq \aleph_\beta < \aleph_\alpha \\ \aleph_{\beta+1} & \text{if } \aleph_\alpha \leq \aleph_\beta \end{cases}$$

Note that even in the presence of GCH, singular cardinals exhibit different behavior: the second clause does not apply if $\aleph_\alpha$ is regular.

3. Consistency and independence of the generalized continuum hypothesis.

The continuum problem remained open until 1939 when a significant progress came from Gödel who showed in [24] that GCH is consistent with the axioms of set theory ZFC (Zermelo-Fraenkel's axioms with the axiom of choice). Gödel produced the model L of *constructible sets* and proved that GCH holds in the model L. In addition to the consistency proof of GCH, Gödel's method introduced the important concept of *inner models*; an interested reader can learn more about this subject in the recent article [34] in this Bulletin.

In 1963, Cohen proved the independence of the continuum hypothesis [7]. Cohen constructed a model of ZFC in which CH fails. Moreover, Cohen's method of *forcing* proved to be a powerful tool for obtaining other independence results, and in particular was used to show that cardinal arithmetic of *regular* cardinals can behave arbitrarily, within the limits imposed by König's Theorem. Shortly after Cohen's breakthrough, Solovay showed that $2^{\aleph_0}$ can take any value not excluded by König's theorem, i.e. one can have $2^{\aleph_0} = \aleph_\alpha$ for any α as long as $\mathrm{cf}\,\aleph_\alpha > \aleph_0$. Then Easton produced a model [12] in which the function $2^{\aleph_\alpha}$, for regular $\aleph_\alpha$, can behave in any prescribed way consistent with König's theorem.

The natural question arose whether the freedom enjoyed by the function $2^{\aleph_\alpha}$ on regular cardinals extends as well to singular cardinals. In particular, can a singular cardinal be the least cardinal at which GCH fails? Or, specifically, is it possible to have $2^{\aleph_n} = \aleph_{n+1}$ for every n while $2^{\aleph_w} = \aleph_{w+2}$? These questions (first asked by Solovay in mid-sixties, see [42]) are a part of what has become known as the *Singular Cardinals Problem*.

At first the consensus among set theorists was that an improvement in Cohen's method will lend to a general consistency result along the lines of Easton's theorem. The truth however turned out to be much more interesting.

4. The singular cardinals problem.

By Easton's theorem, the only rules for the continuum function $2^{\aleph_\alpha}$ on regular cardinals that are provable in ZFC state that $2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$ if $\alpha \leq \beta$, and that cf $2^{\aleph_\alpha} > \aleph_\alpha$. The situation is different for singular cardinals. If $\aleph_\alpha$ is singular then an easy use of cardinal arithmetic shows that $2^{\aleph_\alpha} = (\sum_{\gamma<\alpha} 2^{\aleph_\gamma})^{\mathrm{cf}\,\aleph_\alpha}$. As a consequence, if $\aleph_\alpha$ is singular and if $2^{\aleph_\gamma}$ has the same value $2^{\aleph_{\gamma_0}}$ for eventually all $\gamma < \alpha$ (i.e. $\gamma_0 \leq \gamma < \alpha$) then $2^{\aleph_\alpha} = 2^{\aleph_{\gamma_0}}$ as well. Thus additional rules are needed if one hopes for an extension of Easton's theorem. This was observed e.g. in [3] and [26].

For cardinal arithmetic on singular cardinals, it is not the continuum function $2^{\aleph_\alpha}$ alone that determines all exponentiation. It turns out that it is the function $\aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha}$ that is crucial for cardinal arithmetic. (Note that if $\aleph_\alpha$ is regular then $\aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha} = \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha}$; for singular cardinals we only have $\aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha} \leq 2^{\aleph_\alpha}$.)

The knowledge of the function $\aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha}$ is sufficient for all cardinal exponentiation. That all cardinal arithmetic, including infinite products $\prod$, can be reduced to the function $\aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha}$ was already known to Gödel who in [25] so remarked pointing to Tarski's work [68].

The continuum function is determined as follows: If $\aleph_\alpha$ is regular then $2^{\aleph_\alpha} = \aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha}$. So let $\aleph_\alpha$ be singular. If $2^{\aleph_\gamma} = 2^{\aleph_{\gamma_0}}$ for eventually all $\gamma < \alpha$, then $2^{\aleph_\alpha} = 2^{\aleph_{\gamma_0}}$. If the continuum function $2^{\aleph_\gamma}$ is not eventually constant below $\aleph_\alpha$ then $2^{\aleph_\alpha} = \kappa^{\mathrm{cf}\,\kappa}$ where $\kappa = \sum_{\gamma<\alpha} 2^{\aleph_\gamma}$.

The exponentiation $\aleph_\alpha^{\aleph_\beta}$ is then determined from the continuum function and the function $\kappa^{\mathrm{cf}\,\kappa}$ inductively, the crucial case being when $\mathrm{cf}\,\aleph_\alpha \le \aleph_\beta < \aleph_\alpha$ and $\sum_{\gamma<\alpha} \aleph_\gamma^{\aleph_\beta} = \aleph_\alpha$. Then we have $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\mathrm{cf}\,\aleph_\alpha}$.

Thus the key to cardinal arithmetic is the function $\kappa^{\mathrm{cf}\,\kappa}$. A consequence of König's theorem is that $\kappa^{\mathrm{cf}\,\kappa} \ne \kappa$ and so $\kappa^{\mathrm{cf}\,\kappa} \ge \kappa^+$. Of course if $2^{\mathrm{cf}\,\kappa} \ge \kappa$ then $\kappa^{\mathrm{cf}\,\kappa} = 2^{\mathrm{cf}\,\kappa}$. Hence what is decisive (in addition to the continuum function on regulars) is the behavior of $\kappa^{\mathrm{cf}\,\kappa}$ for those κ for which $2^{\mathrm{cf}\,\kappa} < \kappa$. The simplest possibility is when

$$2^{\mathrm{cf}\,\kappa} < \kappa \text{ implies } \kappa^{\mathrm{cf}\,\kappa} = \kappa^+.$$

This is known as the *Singular Cardinal Hypothesis* (SCH).

Assuming SCH, the continuum function on regular cardinals itself determines cardinal exponentiation: if $\aleph_\alpha$ is singular then $2^{\aleph_\alpha}$ is either κ or κ^+ where $\kappa = \sum\limits_{\gamma<\alpha} 2^{\aleph_\gamma}$, depending on whether $2^{\aleph_\gamma}$ is or is not eventually constant. Similarly for $\aleph_\alpha^{\aleph_\beta}$ where $\mathrm{cf}\,\aleph_\alpha \le \aleph_\beta < \aleph_\alpha$.

With this analysis of cardinal arithmetic it is now clear that the fundamental question related to the singular cardinals problem is whether SCH can fail. In the simplest case mentioned in Section 3, we can ask whether SCH can fail for $\aleph_\omega$: can we have $2^{\aleph_0} < \aleph_\omega$ and at the same time $\aleph_\omega^{\aleph_0} \ge \aleph_{\omega+2}$? (This is a somewhat finer question than whether $\aleph_\omega$ can be a strong limit cardinal while $2^{\aleph_\omega} \ge \aleph_{\omega+2}$; if $2^{\aleph_n} < \aleph_\omega$ for all $\aleph_n$ then $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$.)

Using basic cardinal arithmetic and König's theorem, it is possible to derive several additional rules for the behavior of the function $\kappa^{\mathrm{cf}\,\kappa}$ (see [27]). For instance, if κ is strong limit than $\mathrm{cf}(\kappa^{\mathrm{cf}\,\kappa}) > \kappa$; or if $\kappa \le \lambda^{\mathrm{cf}\,\lambda}$ for some $\lambda < \kappa$ with $\mathrm{cf}\,\lambda \ge \mathrm{cf}\,\kappa$, then $\kappa^{\mathrm{cf}\,\kappa} \le \lambda^{\mathrm{cf}\,\lambda}$. It turns out however that more dramatic restrictions are in store for arithmetic of singular cardinals.

5. Silver's Theorem and Jensen's Covering Theorem.

Until 1974 most set theorists believed that the restriction to regular cardinals in Easton's theorem was due to the weakness of its proof and that analogous results for singular cardinals would be forthcoming. In particular, it was expected that it was possible for a singular cardinal to be the least counterexample to GCH.

This changed dramatically when Silver proved the following theorem [66]:

If κ is a singular cardinal of uncountable cofinality, and if $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

Following Silver's result, several theorems appeared that further restricted the behavior of the function 2^κ for singular cardinals of uncountable cofinality, most notably the theorem of Galvin and Hajnal [14].

If $\aleph_\lambda$ is a strong limit cardinal of uncountable cofinality then $2^{\aleph_\lambda} < \aleph_{(2^\lambda)^+}$.

In another direction, Jensen was able to combine Silver's ideas with his previous work [33] on the fine structure of L, to prove his remarkable Covering Theorem [9]:

If $0^\#$ does not exist then every uncountable set of ordinals can be covered by a constructible set of the same cardinality.

(The existence of $0^\#$ is a large cardinal axiom that we shall return to in the next section.) An easy argument using the Covering Theorem shows that unless $0^\#$ exists, $2^{\mathrm{cf}\,\kappa} < \kappa$ implies $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$, i.e. SCH holds. Thus in order to violate SCH we need large cardinals.

A related corollary of the Covering Theorem states that if $0^\#$ does not exist then for every $\lambda \geq \aleph_2$, if λ is a regular cardinal in L then $\mathrm{cf}\ \lambda = |\lambda|$. In particular, a regular cardinal cannot be changed into a singular cardinal in the absence of large cardinals. (The assumption $\lambda \geq \aleph_2$ is necessary, as Bukovský [4] and Namba [45] produced a forcing extension of L — therefore $0^\#$ does not exist — in which $|\omega_2^L| = \aleph_1$ and $\mathrm{cf}\,\omega_2^L = \omega$.)

The assumption of Silver's Theorem that κ has uncountable cofinality figures prominently in the proof. To let the reader appreciate the significance of uncountable cofinality, I shall give a brief outline of the methods used in Silver's and Galvin-Hajnal's theorems.

For simplicity, let us consider $\aleph_{\omega_1}$. For Silver's Theorem, assume that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$. The main idea of the proof is that there exists a generic extension $V[G]$ of the universe, in which there exists a *normal* ultrafilter U on $P^V(\omega_1)$. Normality means, as usual, that the diagonal is the least nonconstant function; here the fact that ω_1 is uncountable is essential. Working in $V[G]$, we can calculate the size of $(2^{\aleph_{\omega_1}})^V$ as follows. Consider the ultrapower M of V by U, and the elementary embedding j: $V \to M$. M is not well founded, but thanks to normality, the M-ordinal κ represented by the function $\alpha \mapsto \aleph_\alpha$ is the supremum of the M-ordinals $j(\aleph_\gamma)$, $\gamma < \omega_1$. It follows that the linearly ordered set κ^+ has size at most $((\aleph_{\omega_1})^V)^+$. A further argument gives $|P^V(\aleph_{\omega_1})| \le |P^M(\kappa)|$, and since $M \models 2^\kappa = \kappa^+$, we have $|(2^{\aleph_{\omega_1}})^V| \le ((\aleph_{\omega_1})^V)^+$. This is calculated in $V[G]$, but since the forcing extension is "mild", the same inequality holds in V.

Silver's proof has been reworked in [1], replacing the forcing technique by a purely combinatorial argument. A further improvement of the combinatorial method led to the above mentioned theorem of Galvin and Hajnal. In the Galvin-Hajnal theorem, the uncountability of the cofinality is again essential. Roughly speaking, the generic ultrapower of Silver is replaced by a well founded partial ordering of ordinal-valued functions. If f and g are ordinal functions on ω_1, let $f < g$ denote the relation

$$\{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \text{ contains a club.}$$

Due to normality of the club filter, the relation $<$ is a well founded partial ordering. Hence every ordinal function f on ω_1 can be assigned its rank in $<$, the *Galvin-Hajnal norm* $\|f\|$ of f. The essence of Galvin-Hajnal's proof is that the size of $2^{\aleph_{\omega_1}}$ (if $\aleph_{\omega_1}$ is strong limit) is related to the Galvin-Hajnal norm of the function $\alpha \mapsto 2^{\aleph_\alpha}$. The analysis of this relationship yields the upper bound $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

6. Large cardinals.

As the singular cardinals problem is so closely related to large cardinal axioms, we shall now review the basics of the theory of large cardinals.

An uncountable cardinal number κ is *inaccessible* if it is regular and a strong limit cardinal. An immediate consequence of inaccessibility is that V_κ, the collection of all sets of rank less than κ, is a model of ZFC; another immediate consequence is that $\kappa = \aleph_\kappa$ is a fixed point of the aleph sequence. By Gödel's 2nd Incompleteness Theorem it follows that the existence of inaccessible cardinals is unprovable in ZFC. In fact, a slightly more involved argument shows that the relative consistency of inaccessible cardinals is unprovable. Thus the existence of inaccessibles is to ZFC as the existence of an infinite set is to Peano arithmetic. For that reason, large cardinal axioms are sometimes referred to as *strong axioms of infinity*.

Modern large cardinal theory recognizes a substantial number of large cardinal axioms. Interestingly enough, these axioms form a linearly ordered scale, on which the relation of a stronger axiom to the weaker theories is just as described above in the case of inaccessible cardinals and ZFC. This scale of large cardinals serves as a measure of consistency strength of various set theoretic assumptions (including assumptions on singular cardinals).

One of the most prominent large cardinals is a measurable cardinal. Measurable cardinals were introduced by Ulam while working on the *measure problem*, the question whether some set E can carry a nontrivial countably additive measure such that every subset of E is measurable. An uncountable cardinal κ is *measurable* if κ carries an atomless κ-additive 2-valued measure; equivalently, if κ carries a nonprincipal κ-complete ultrafilter. (The terms "κ-additive" and "κ-complete" refer to unions and intersections of *fewer than* κ sets.) In [49], Scott applied to measurable cardinals the method of ultraproducts, thus laying the groundwork for the modern large cardinal theory. One of the basic observations is that the existence of measurable cardinals is equivalent to the existence of a nontrivial elementary embedding $j : V \to M$ where M is some transitive model. Every measurable cardinal is not only inaccessible, but on the scale of large cardinals is above Mahlo, weakly compact and Ramsey cardinals, to mention just the most important categories of large cardinals (see picture).

Scott's theorem [49] states that measurable cardinals don't exist in L. The subsequent work of Gaifman, Rowbottom [48] and Silver [65] established the strength of measurable cardinals *vis à vis* constructible sets, isolating the principle "$0^{\#}$ exists". The existence of $0^\#$ (between weakly compact and Ramsey on our scale) states that there aren't many constructible sets, and the truth definition for the model L exists. Equivalently, there exists a nontrivial elementary embedding $j : L \to L$. One consequence of $0^\#$ is that *every* uncountable cardinal (in V) is inaccessible in L. In particular, $\aleph_\omega$ is regular in L; recall that by the Covering Theorem, if $0^\#$ does not exist then $\aleph_\omega$ (and every singular cardinal) must be singular in L.

Among large cardinals stronger than measurable, *supercompact* cardinals are most prominent. There exists at present a good classification of large cardinals between measurables and *Woodin cardinals*, using the theory of *inner models* (we refer the reader to Jensen's forthcoming article [34] in this Bulletin). In particular, measurable cardinals are classified by their *order* $o(\kappa)$, where $o(\kappa) = 1$ means that κ is measurable, $o(\kappa) = 2$ means that κ carries a normal measure in which almost all $\alpha < \kappa$ are measurable, and so on, up to $o(\kappa) = \kappa^{++}$.

Above supercompacts, the scale approaches its end with the existence of an elementary embedding $j : V_\lambda \to V_\lambda$, as by a theorem of Kunen, $j : V \to V$ is inconsistent.

For the benefit of nonspecialists, as well as a reference for the forthcoming sections we present a scale of the more prominent large cardinals:

7. Large cardinals and the singular cardinals problem.

Prior to Jensen's Covering Theorem there had been scattered results indicating that there might be some link between the behavior of singular cardinals and the large cardinal axioms. For instance, Solovay proved in [67] that if κ is a supercompact cardinal and $\lambda > \kappa$ is singular then $\lambda^{\operatorname{cf}\lambda} = \lambda^+$. This means that the SCH holds above the least supercompact cardinal. On the other hand, Prikry introduced in [46] a method of forcing that changes the cofinality of a measurable cardinal to ω without collapsing it. The measurable cardinal κ remains a cardinal but has $\operatorname{cf}\kappa = \omega$ in the extension. Subsequently, Silver devised a forcing method that, starting with a supercompact cardinal κ, produced a model in which κ is measurable and $2^\kappa > \kappa^+$. This, combined with the Prikry forcing, yields the consistency of the negation of SCH, relative to a supercompact cardinal: a model of ZFC in which κ is a strong limit cardinal, $\operatorname{cf}\kappa = \omega$, and $2^\kappa > \kappa^+$.

About the same time that Jensen established, by the Covering Theorem, the necessity of large cardinals for the negation of the SCH, Magidor obtained the first in a series of consistency results using large cardinals. In [39], he proved the consistency of $2^{\aleph_\omega} > \aleph_{\omega+1}$ (and $\aleph_\omega$ strong limit) from a supercompact cardinal, and in [40] the consistency of $2^{\aleph_\omega} = \aleph_{\omega+2}$ along with GCH below $\aleph_\omega$, from a 2-huge cardinal. When it soon became clear that large cardinals are indeed necessary, Magidor's method was refined to yield other consistency results. The general idea in all these proofs is as follows: start with a large cardinal κ, blow up 2^κ, change the cofinality of κ by adjoining a new set of cardinals C cofinal in κ, and destroy all cardinals below κ that are not in the set C. The technique employed to change the cofinality evolved from Prikry's method, progressing through [41] to [47]. More recently, a new method for building models for the negation of SCH was developed in [19] and [21].

In Magidor's model [39], $2^{\aleph_\omega}$ could be as large as $\aleph_{\omega+\omega+1}$, which was improved by Shelah [55] as to have $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for any $\alpha < \omega_1$. Woodin [13] constructed a model in which $2^\kappa = \kappa^{++}$ for all infinite cardinals κ, and Cummings [8] found a model in which GCH fails exactly at all limit cardinals.

As for the consistency strength of the failure of SCH, a series of results proved it to be exactly a measurable cardinal of order $o(\kappa) = \kappa^{++}$. Magidor's assumption of supercompactness was first weakened by Woodin, and eventually Gitik obtained a model of SCH in [16] using $o(\kappa) = \kappa^{++}$. The other direction, namely that $o(\kappa) = \kappa^{++}$ is a necessary assumption, is also due to Gitik ([17]), and uses the technique of inner models.

The technique of inner models is an outgrowth of Jensen's Covering Theorem. The first step up from $0^{\#}$ was the core model K and the covering theorem for K ([10], [11]), followed by inner models and covering theorems for measurable cardinals [43] and beyond. Details can be found in Jensen's article [34]. This technique produced a number of results, notably [44], [20], [18] and [22] showing the necessity of large cardinal axioms for various violations of SCH.

A major open problem in cardinal arithmetic is whether $2^{\aleph_\omega}$ can be greater than $\aleph_{\omega_1}$ (while $\aleph_\omega$ is strong limit). The analysis provided by the pcf theory indicates that an entirely new approach would be needed, and the inner model technique shows that its consistency strength is enormous.

8. Upper bounds.

Following Silver's Theorem [66], one of the main directions of research in the theory of singular cardinals has been the search for upper bounds on $2^{\aleph_\alpha}$ for strong limit singular cardinals $\aleph_\alpha$; or more precisely, on the function $\aleph_\alpha^{\operatorname{cf}\aleph_\alpha}$ for those cardinals that satisfy $2^{\operatorname{cf}\aleph_\alpha} < \aleph_\alpha$. Let us recall that the Galvin-Hajnal Theorem gives a bound

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$$

if $\aleph_{\omega_1}$ is a strong limit cardinal. In full generality, if $\aleph_\delta$ is a strong limit cardinal of uncountable cofinality then $2^{\aleph_\delta} < \aleph_{(|\delta|^{\operatorname{cf}\delta})^+}$.

The theorem leaves open the question whether an upper bound exists for $2^{\aleph_\omega}$ (and other cardinals of cofinality ω). The method does not give any information in this case, as it depends heavily on the use of closed unbounded sets.

The other question left open is the case when $\aleph_\delta$ is a fixed point of the aleph function, i.e. when $\delta = \aleph_\delta$.

Several papers extended the Galvin-Hajnal result under large cardinal assumptions: Magidor [38] proved $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$ from Chang's Conjecture (which has the consistency strength of approximately Ramsey cardinals); Jech and Prikry [30] proved $2^{F(\aleph_1)} < F(\aleph_2)$, where $F(\aleph_\alpha)$ is the $\aleph_\alpha th$ fixed point of the $\aleph$ function, from a saturated ideal (approximately supercompact); and Shelah [53] proved the same result from Chang's Conjecture. In [54] and [56] Shelah eliminated the large cardinal assumptions, obtaining somewhat weaker bounds, such as $2^{F(\aleph_1)} < F((2^{2^{\aleph_1}})^+)$. Interestingly, his method broke down for the $\omega^{\rm th}$ iteration of the fixed point operation; eventually it turned out that in this case no a priori bounds are provable.

To obtain upper bounds for $2^{\aleph_\omega}$, a different approach is needed. The major breakthrough came when Shelah discovered the significance of cofinalities of ultraproducts $\prod_{n=0}^\infty \aleph_n/D$, where D is an ultrafilter on ω. Making use of these, he obtained the analog of the Galvin-Hajnal Theorem: if $\aleph_\omega$ is strong limit then

$$2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$$

(cf. [50], Chapter XIII). The proof of this theorem led Shelah to a systematic study of cofinalities of reduced products of sets of cardinals. In a sequence of papers [53], [57], [58] and [51] (Chapters II, VIII and IX), he developed a beautiful theory that brought a number of unexpected results and yielded deep applications to cardinal arithmetic. The crowning achievement so far is the following result that we shall discuss in the following section: if $2^{\aleph_0} < \aleph_\omega$, then

$$\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}.$$

9. Shelah's pcf theory.

In this last section we shall outline Shelah's theory of possible cofinalities (pcf). The central concept of the theory is the cofinality of an ultraproduct of a set of regular cardinals:

Let A be a set of regular cardinals, and let D be an ultrafilter on A. The ultraproduct $\prod A/D$ consists of equivalence classes of ordinal functions f such that $\mathrm{dom}(f) = A$ and $f(\alpha) \in \alpha$ for all $\alpha \in A$; two functions f and g are equivalent mod D if $\{\alpha : f(\alpha) = g(\alpha)\} \in D$. The ultraproduct is linearly ordered by the relation $f <_D g$ defined by $\{\alpha : f(\alpha) < g(\alpha)\} \in D$, and the *cofinality* of $\prod A/D$ is the unique regular cardinal $\kappa = \mathrm{cof} \prod A/D$ such that $(\prod A/D, <_D)$ has a cofinal subset of order type κ.

If $\kappa = \mathrm{cof} \prod A/D$ for some ultrafilter D, we say that κ is a *possible cofinality* (of $\prod A$), and we let

$$\mathrm{pcf}\, A = \{\mathrm{cof} \prod A/D : D \text{ an ultrafilter on } A\}$$

be the set of all possible cofinalities of $\prod A$.

As a typical case, let us consider the set $A = \{\aleph_n\}_{n=0}^{\infty}$. Of course, every $\aleph_n$ is a possible cofinality, by a principal ultrafilter. If $\kappa = \mathrm{cof} \prod A/D$ where D is nonprincipal then clearly $\kappa \geq \aleph_{\omega+1}$, and since $|\prod A| = \aleph_\omega^{\aleph_0}$, we have $\kappa \leq \aleph_\omega^{\aleph_0}$. So unless $\aleph_\omega^{\aleph_0} > \aleph_{\omega+1}$, the set pcf A has only one element outside A, and so a meaningful theory of possible cofinalities requires the negation of the singular cardinals hypothesis. It turns out that the pcf theory is more fundamental than cardinal arithmetic.

In [52] Shelah proved, among others, the following result that was a precursor of the future pcf theory:

Theorem 9.1. *For every singular cardinal λ of cofinality ω there exists an increasing sequence $\{\lambda_n\}_{n=0}^{\infty}$ of regular cardinals with limit λ such that cof $\prod_{n=0}^{\infty} \lambda_n/D = \lambda^+$ for every nonprincipal ultrafilter D.*

This result shows, e.g., that $\aleph_{\omega+1}$ is a possible cofinality of $\prod_{n=0}^{\infty} \aleph_n$ regardless of the value of $\aleph_\omega^{\aleph_0}$.

Shortly thereafter, Shelah established the relation between possible cofinalities and cardinal arithmetic, and obtained an upper bound for $2^{\aleph_\omega}$, cf. [50]. We say that A is an *interval* of regular cardinals if whenever $\alpha < \beta < \gamma$ are regular cardinals and $\alpha, \gamma \in A$, then $\beta \in A$. The following theorem reduces the simplest cases of the singular cardinals problem to the problem of possible cofinalities:

Theorem 9.2. *(a) If A is an interval of regular cardinals such that $|A| < \min A$ then pcf A is an interval.*

(b) If, moreover, $(\min A)^{|A|} < \sup A$ then $\kappa = |\prod A|$ is a possible cofinality.

This immediately gives an upper bound on $2^{\aleph_\omega}$ if $\aleph_\omega$ is strong limit: since $|\mathrm{pcf}\{\aleph_n\}_{n=0}^{\infty}| \leq 2^{2^{\aleph_0}}$ (the number of ultrafilters on a countable set), and since we assume that $2^{2^{\aleph_0}} < \aleph_\omega$, we have $2^{\aleph_\omega} = |\prod_{n=0}^{\infty} \aleph_n| < \aleph_{\aleph_\omega}$. It also reduces the problem of evaluating $\aleph_\omega^{\aleph_0}$ (and similar products) to finding the maximum of possible cofinalities of $\prod_{n=0}^{\infty} \aleph_n$. (There are analogous reductions for cases not covered by Theorem 9.2.) Notice another consequence of Theorem 9.2: as $|\prod A|$ is a possible cofinality, it must be a regular cardinal. Hence if $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega}$ is regular. This does not follow from König's Theorem.

If $2^{\aleph_0} < \aleph_\omega$ then all regular cardinals between $\aleph_{\omega+1}$ and $\aleph_\omega^{\aleph_0}$ are possible cofinalities of $\prod_{n=0}^{\infty} \aleph_n$. It is interesting to see how they are attained in the models where $2^{\aleph_\omega} > \aleph_{\omega+1}$. A detailed analysis of this was done in [28] for Magidor's model [39]; a similar analysis is possible for other models. Let κ_n, $n < \omega$, be the cardinals in the Prikry sequence; e.g., in the model where $2^{\aleph_\omega} = \aleph_{\omega+2}$ these are $\kappa_n = \aleph_{3n}$. Then cof $\prod_{n<\omega} \kappa_n^+/D = \aleph_{\omega+1}$ and cof $\prod_{n<\omega} \kappa_n^{++}/D = \aleph_{\omega+2}$, for every nonprincipal ultrafilter. Hence $\prod_n \aleph_{3n+1}$ has cofinality $\aleph_{\omega+1}$ while $\prod_n \aleph_{3n+2}$ has cofinality $\aleph_{\omega+2}$!

Fundamentals of the pcf theory. The theory developed by Shelah in [52], [57] and [58], and described in detail in the monograph [51] analyzes possible cofinalities of products $\prod A$ where A is a set of regular cardinals with the property that $|A| < \min A$. (For rather trivial reasons, the general pcf theory breaks down in the case when $\sup A$ is a fixed point of the aleph function.) We shall now present the main points of the theory; the reader interested in proofs of the theorems stated here and in the techniques involved might find either [5] or [29] useful.

Let A be a set of regular cardinals and assume that $|A| < \min A$, and let

$$\text{pcf } A = \{\text{cof } \prod A/D : D \text{ an ultrafilter on } A\}.$$

First we mention the trivial facts about pcf:

$$A \subseteq \text{pcf } A,$$

$$\text{if } A_1 \subseteq A_2 \text{ then pcf } A_1 \subseteq \text{pcf } A_2,$$

$$\text{pcf}(A_1 \cup A_2) = \text{pcf } A_1 \cup \text{pcf } A_2.$$

Another not so difficult observation is that when $|\text{pcf } A| < \min A$ then pcf pcf $A =$ pcf A.

The key result of the pcf theory is the following:

Theorem 9.3. *(Existence of generators.) There exists a collection $\{B_\nu : \nu \in pcf\ A\}$ of subsets of A such that for every ν,*

(i) $\nu = \max pcf\ B_\nu$,

(ii) $\nu \notin pcf(A - B_\nu)$.

(The sets B_ν are called *generators* of pcf A.)

In other words:

(i) for every ultrafilter D on B_ν, cof $\prod A/D \leq \nu$; and there exists some D on B_ν such that cof $\prod A/D = \nu$,

(ii) for every D, if cof $\prod A/D = \nu$ then $B_\nu \in D$.

Therefore, the cofinality of $\prod A/D$ is determined by which generators belong to D :

$$\text{cof } \prod A/D = \text{least } \lambda \text{ such that } B_\lambda \in D.$$

Let us note some consequences of Theorem 9.3:

(iii) $|\text{pcf } A| \leq 2^{|A|}$.

This is because $|\text{pcf } A|$ does not exceed the number of generators which is at most $2^{|A|}$. An immediate consequence is a better bound on $2^{\aleph_\omega}$ if $\aleph_\omega$ is a strong limit: $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.

(iv) pcf A has a maximal element.

If not, then $\{A - B_\nu : \nu \in \text{pcf } A\}$ has the finite intersection property; let D be an ultrafilter extending it, and let $\lambda = \text{cof } \prod A/D$. Then $B_\lambda \in D$, a contradiction.

The same argument proves the following important property of generators:

Theorem 9.4. *(Compactness.) For every $X \subseteq A$ there exists a finite set of cardinals $\nu_1, ..., \nu_k \in \text{pcf } X$ such that $X \subseteq B_{\nu_1} \cup \cdots \cup B_{\nu_k}$.*

Transitive generators and the localization theorem. Generators of pcf A are not necessarily unique, as (e.g.) changing B_ν by a finite set won't affect properties (i) or (ii). The B_ν's are however unique up to the B_η's for $\eta < \nu$. More precisely: for every $\kappa \leq \max \text{pcf } A$, let

$$J_\kappa = \text{ the ideal (of subsets of } A\text{) generated by the sets } B_\eta, \text{ for } \eta < \kappa \text{ and } \eta \in \text{pcf } A.$$

Properties (i) and (ii) imply that for every $X \subseteq A$,

$$X \in J_\kappa \text{ if and only if cof } \prod X/D < \kappa \text{ for every ultrafilter } D \text{ on } X,$$

and that each generator B_ν is unique mod J_ν.

An important feature of the pcf theory is that pcf A has *transitive* generators (incidentally, this theorem is highly nontrivial):

Theorem 9.5. *There exist generators B_ν, $\nu \in \text{pcf } A$, such that whenever $\eta \in B_\nu$ then $B_\eta \subseteq B_\nu$.*

Transitivity of generators is particularly useful when appied to sets $A = \text{pcf } A$. A fairly straightforward use of transitivity and compactness yields the following important result:

Theorem 9.6. *(Localization Theorem.) Let $X \subseteq \text{pcf } A$ and let $\lambda \in \text{pcf } X$. Then X has a subset X_0 such that $|X_0| \leq |A|$, and such that $\lambda \in \text{pcf } X_0$.*

An upper bound for $|\text{pcf } A|$. The strength of the Localization Theorem is best illustrated by its application providing an upper bound on the size of pcf A, and consequently, on the function $2^{\aleph_\alpha}$ for singular $\aleph_\alpha$. We shall now outline this application of the pcf theory.

First we should mention that one application of the techniques used in the pcf theory is the following analog of Theorem 9.1 for uncountable cofinality:

Theorem 9.7. *If λ is a singular cardinal of uncountable cofinality then there exists a closed unbounded set $C \subseteq \lambda$ such that cof $\prod_{\alpha \in C} \alpha^+/D = \lambda^+$ for every ultrafilter D concentrating on end-segments of C.*

Now let $A = \{\aleph_n\}_{n=0}^\infty$. By theorem 9.2, pcf A is an interval of regular cardinals and, if $2^{\aleph_0} < \aleph_\omega$ then max pcf $A = \aleph_\omega^{\aleph_0}$. We claim that max pcf $A < \aleph_{\omega_4}$, and therefore $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$.

The structure of pcf A induces a closure operation on Θ, where $\aleph_{\Theta+1} = \max\ \text{pcf } A$: if $X \subseteq \Theta$, we let

$$\alpha \in \overline{X} \text{ if and only if } \aleph_{\alpha+1} \in \text{pcf } \{\aleph_{\xi+1} : \xi \in X\}.$$

From Theorem 9.7 it follows that if $\lambda < \Theta$ is an ordinal of uncountable cofinality then there exists a club $C \subseteq \lambda$ such that $\sup \overline{C} = \lambda$. From the Localization Theorem it follows that every set $X \subseteq \Theta$ of order type ω_1 has a countable initial segment X_0 such that $\sup \overline{X_0} \geq \sup X$.

Shelah now proceeds to show that if $\Theta \geq \aleph_4$ then no closure operation on Θ with those two properties exists. Therefore $\Theta < \aleph_4$, and $\max\ \text{pcf}\{\aleph_n\}_{n=0}^\infty < \aleph_{\omega_4}$!

There are some elements of the proof that are specific to $\aleph_4$, making it impossible at present to bring down the upper bound to, say, $\aleph_{\omega_3}$. The method definitely does not work for $\aleph_{\omega_1}$, cf. [32], but it is still an open problem whether $2^{\aleph_\omega}$ can be greater than $\aleph_{\omega_1}$ (with $\aleph_\omega$ strong limit). In fact, it is still unknown whether $|A| < |\text{pcf } A|$ is possible (together with $|A| < \min A$).

Reduced products of ordinals. The main technique used in the analysis of possible cofinalities involves reduced products of ordinal numbers. We shall conclude the article with a brief description of this topic.

Let A be an infinite set, and let I be an ideal on A. For ordinal functions f, g on A we define

$$f =_I g \quad \text{if} \quad \{a \in A : f(a) \neq g(a)\} \in I,$$

$$f <_I g \quad \text{if} \quad \{a \in A : f(a) \geq g(a)\} \in I.$$

Let λ_a, $a \in A$, be limit ordinal numbers. The *reduced product*

$$\prod_{a \in A} \lambda_a / I$$

consists of equivalence classes of ordinal functions f mod $=_I$, such that $f(a) \in \lambda_a$ for all $a \in A$.

Let κ be a regular cardinal. We say that $P = \prod_{a \in A} \lambda_a / I$ is $\kappa-$*directed* if every subset X of P of size $< \kappa$ has an upper bound in P, i.e. a function $f \in P$ such that $g <_I f$ for every $g \in X$. We say that P has *true cofinality* κ if there exists an increasing sequence $\langle f_\alpha : \alpha < \kappa \rangle$ cofinal in P, i.e. $f_0 <_I f_1 <_I \ldots$, and for every $g \in P$ there is an α such that $g <_I f_\alpha$. Note that if P has true cofinality κ then P is $\kappa-$directed.

We recall that for a given set A of regular cardinals, J_κ ($\kappa \leq \max\ \text{pcf}\ A$) denotes the ideal on A generated by the B_ν, $\nu < \kappa$. The crucial property of the generators which enables the analysis of pcf A is this:

Theorem 9.8. *If A is a set of regular cardinals such that $|A| < \min A$ and if $\kappa \in \text{pcf}\ A$ then $\prod_{\nu \in B_\kappa} \nu / J_\kappa$ has true cofinality κ.*

In fact, the technology provided by the pcf theory goes beyond analyzing the ideals J_κ. Perhaps the most general result on reduced products is the following. (This is explicitly stated for $\kappa > 2^{|A|}$ in Lemma 2.4 of [29], and can be proved for $\kappa > |A|^+$ as in Theorems 7.3 and 7.9 of [5]):

Theorem 9.9. *(Splitting Theorem.) Let λ_a, $a \in A$, be limit ordinals of cofinality greater than $|A|^+$ and let κ be a regular cardinal, $\kappa > |A|^+$. If $\prod_{a\in A} \lambda_a/I$ is $\kappa-$directed, then either $\prod_{a\in A} \lambda_a/I$ is κ^+-directed, or it has true cofinality κ, or there exist $A_1, A_2 \notin I$, $A_1 \cup A_2 = A$, such that $\prod_{a\in A_1} \lambda_a/I$ is κ^+-directed and $\prod_{a\in A_2} \lambda_a/I$ has true cofinality κ.*

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