SOME RESULTS ON COMBINATORS IN THE SYSTEM TRC

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Abstract. We investigate the system TRC of untyped illative combinatory logic that is equiconsistent with New Foundations. We prove that various unstratified combinators do not exist in TRC.

Introduction. We prove some results in the axiomatic system TRC introduced in [3]. The system TRC (for 'type-respecting combinators') is an untyped system of combinatory logic, in the sense of [1], [2]. TRC is a first order theory of functions (combinators) with equality and is illative, i.e. capable of expressing notions of propositional logic. Moreover, it is combinatorially complete for stratified combinators. The main interest of TRC is that it is equiconsistent with the theory NF [6], Quine's 'New Foundations'. As the consistency of NF remains an open problem, so does the consistency of TRC.

The objects of study of a combinatory logic are *combinators*. We denote xy the application of the combinator x to the combinator y, and adopt the convention that $xyz = (xy)z$.

The language of TRC has (in addition to equality and the binary function xy) constants $Abst$, Eq, p_1 and p_2, and functions $k(x)$ and $\langle x, y\rangle$. The axioms of TRC are the following:

I. $k(x)y = x$.

II. $p_i\langle x_1, x_2\rangle = x_i$ for $i = 1, 2$.

III. $\langle p_1 x, p_2 x\rangle = x$.

IV. $\langle x, y\rangle z = \langle xz, yz\rangle$

V. $Abst\, x\, y\, z = x\, k(z)(yz)$.

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VI. $Eq\langle x,y\rangle = p_1$ if $x = y$; $Eq\langle x,y\rangle = p_2$ if $x \neq y$.

VII. If for all z, $xz = yz$, then $x = y$.

VIII. $p_1 \neq p_2$.

Axiom I postulates the existence of constant functions. Axioms II–IV describe the pairing function $\langle x,y\rangle$ and the projections p_1 and p_2. *Abst* is the abstraction combinator and Eq is the characteristic function of equality. Axiom VII is the axiom of extensionality.

Let $I = \langle p_1, p_2\rangle$; from III and IV it follows that I is the identity function $Ix = x$.

Classical combinatory logic [2,3] employs combinators I, K and S, where

$$Ix = x \qquad Kxy = x \qquad Sxyz = xz(yz).$$

It has a powerful abstraction property: for every term t and a variable x, there is a term λxt in which x does not occur, such that for every term s,

$$(\lambda xt)s = t[s/x].$$

This guarantees, among others, the existence of a fixed point for every combinator, and implies that simple notions of propositional logic cannot be represented by combinators. Suppose that Neg is the negation combinator, and consider $u = \lambda x(Neg(xx))$. Then $uu = Neg(uu)$.

The theory TRC is an illative theory, in the sense that it can encode notions of propositional logic. It also has an abstraction property (Theorem 1 of [3]). The term λxt can be constructed for every t in which x occurs with no type other than 0. (For details about typing see [3].) It follows that TRC proves the existence of all *stratified* combinators. Examples of stratified combinators are $x(xy)$, $xy(yz)$, $y(xyz)$: in $y(xyz)$, z has type 0, y has type 1 and x has type 2. (In fact, $Abst\,Abst\,Ixy = x(xy)$, $Abst(AbstAbst)xyz = xy(yz)$, and $AbstAbst\,xyz = y(xyz)$).

We will show in Section 3 that (with the exception of I) the standard combinators used in classical combinatory logic do not exist in TRC. We shall give many examples of unstratified combinators whose existence contradicts the axioms of TRC.

In searching for proofs of the various results in TRC, we used a computer extensively and used the automated theorem prover OTTER [5].

2. Some Basic Facts on TRC.

In this sectioin we derive some simple equalities from the axioms of TRC, and use a self-reference argument to obtain some simple negative results. First we state some properties of the abstraction combinator (see also [4]):

Theorem 2.1. (a) $Abst(Abst(Abst\ x)) = Abst\, x$.

(b) $Abst(Abst\, k(x)) = k(x)$.

(c) $Abst\ k(k(x)) = k(k(x))$.

(d) $Abst\ k(x)yz = x(yz)$.

(e) $Abst\ k(x)k(y) = k(xy)$.

Proof. The equalities are obtained by an application of the axioms defining $Abst$ and $k(x)$ and the axiom of extensionality; e.g. to prove (a), we evaluate the term $Abst(Abst(Abst\ x))yz$ and compare it with $Abst\ x\, y\, z$.

□

The next theorem gives some properties of the pairing function and the projections:

Theorem 2.2. (a) $\langle k(x), y\rangle z = \langle x, yz\rangle, \langle x, k(y)\rangle z = \langle xz, y\rangle$.

(b) $k(\langle x, y\rangle) = \langle k(x), k(y)\rangle,\ k(p_i x) = p_i k(x)$ *for* $i = 1, 2$.

(c) $p_i(xy) = p_i xy$ *for* $i = 1, 2$.

Proof. (a) From Axiom IV.

(b) By Axiom IV, $\langle k(x), k(y)\rangle z = \langle k(x)z, k(y)z\rangle = \langle x, y\rangle = k(\langle x, y\rangle)z$, and by extensionality, $k(\langle x, y\rangle) = \langle k(x), k(y)\rangle$. Using this and Axiom III, we get $k(x) = k(\langle p_1 x, p_2 x\rangle) = \langle k(p_1 x), k(p_2 x)\rangle$, and so by Axiom II, $p_i k(x) = k(p_i x)$.

(c) Using Axioms III, IV and II, we get $p_i(xy) = p_i(\langle p_1 x, p_2 x\rangle y) = p_i\langle p_1 xy, p_2 xy\rangle = p_i xy$. □

Next we state some more properties of the combinator $Abst$:

Theorem 2.3. (a) $Abst\langle x, y\rangle = \langle Abst\ x, Abst\ y\rangle$.

(b) $Abst\ p_i = k(p_i)$ *and* $Abst\ k(p_i) = p_i$, *for* $i = 1, 2$.

(c) $Abst\ I = k(I)$ *and* $Abst\ k(I) = I$.

Proof. (a) Using Axiom IV, show that $Abst\langle x, y\rangle uv = \langle Abst\ x, Abst\ y\rangle uv$.

(b) $Abst\ p_i xy = p_i k(y)(xy) = k(p_i y)(xy) = p_i y$ by Theorem 2.2b, and $Abst\ k(p_i)xy = p_i(xy) = p_i xy$ by Theorems 2.1d and 2.2c.

(c) $Abst\, Ixy = y = k(I)xy$, by Axioms V and I, and $Abst\ k(I)xy = I(xy) = Ixy$ by Theorem 2.1d. □

We shall now turn to negative results. In Section 3 we shall present a number of combinators that do not exist in TRC. Each proof will use one of the following basic negative results that use self-reference:

Theorem 2.4. *For every x,*

(a) $Eq\langle x, p_2\rangle \neq x$.
(b) $Eq\langle k(x), k(p_2)\rangle \neq x$.
(c) $\langle Eq\, x, p_2\rangle \neq x$.

Proof. (a) $Eq\langle x, y\rangle$ is either p_1 or p_2, and $Eq\langle p_1, p_2\rangle = p_2$ while $Eq\langle p_2, p_2\rangle = p_1$.

The proof of (b) and (c) is similar. □

It follows from the discussion on classical combinatory logic in Section 1 that not every combinator in TRC has a fixed point. Theorem 2.4 gives an explicit example, $\langle Eq, k(p_2)\rangle x \neq x$:

Corollary 2.5. *The combinator $\langle Eq, k(p_2)\rangle$ does not have a fixed point.*

A standard fact of combinatory logic (cf. [1], [2]) states that if M is the combinator $Mx = xx$ then for every u, the composition of u and M is a fixed point of u. As the abstraction theorem for TRC in [3] provides for composition of combinators, it follows that M does not exist in TRC. Here we give a direct proof:

Theorem 2.6. *There is no M such that $Mx = xx$.*

Proof. Let $t = \mathit{Abst}\, k(Eq)\langle M, k(p_2)\rangle$ and let $s = tt$. Then (using Theorems 2.1.d and 2.2a)

$$\begin{aligned} s = tt &= \mathit{Abst}\, k(Eq)\langle M, k(p_2)\rangle t \\ &= Eq(\langle M, k(p_2)\rangle t) \\ &= Eq\langle Mt, p_2\rangle \\ &= Eq\langle tt, p_2\rangle \\ &= Eq\langle s, p_2\rangle, \end{aligned}$$

contradicting Theorem 2.4a. □

A similar argument, using Theorem 2.4b, yields the following:

Theorem 2.7. *There is no K_1 such that $K_1 x = k(xx)$.*

Proof. Let $t = \mathit{Abst}\, k(Eq)\langle K_1, k(k(p_2))\rangle$, and $s = tt$. Then (by Theorems 2.1d and 2.2a)

$$\begin{aligned} s = tt &= \mathit{Abst}\, k(Eq)\langle K_1, k(k(p_2))\rangle t \\ &= Eq(\langle K_1, k(k(p_2))\rangle t) \\ &= Eq\langle K_1 t, k(p_2)\rangle \\ &= Eq\langle k(s), k(p_2)\rangle, \end{aligned}$$

contradicting Theorem 2.4b. □

An immediate consequence of Theorem 2.7 is that neither $k(x)$ nor $\langle x, y\rangle$ can be replaced in TRC by a combinator (see also [4]).

Theorem 2.8. (a) *There is no K such that $Kx = k(x)$.*

(b) *There is no p such that $pxy = \langle x, y\rangle$.*

Proof. (a) Given such K, let $K_1 = \mathit{Abst}\mathit{Abst}\ K$. Then (by Theorem 2.1d)

$$\begin{aligned} K_1xy &= \mathit{Abst}\mathit{Abst}\ Kxy \\ &= \mathit{Abst}\ K(x)(Kx)y \\ &= x(Kxy) \\ &= xx \end{aligned}$$

and so $K_1x = k(xx)$, contradicting Theorem 2.7.

(b) Given p, let $K = p_1p$, and then (by Theorem 2.2c)

$$\begin{aligned} Kxy &= p_1pxy \\ &= p_1(px)y \\ &= p_1(pxy) \\ &= p_1\langle x, y\rangle \\ &= x, \end{aligned}$$

contradicting (a). □

We conclude this section with the following result that we use in Section 3.

Theorem 2.9. (a) *There is no u such that $ux = xk(x)$.*

(b) *There is no u such that $uk(x) = xk(x)$.*

Proof. (a) Given u, let $M = \mathit{Abst}(\mathit{Abst}\ u)I$, and then

$$\begin{aligned} Mx &= \mathit{Abst}(\mathit{Abst}\ u)I\, x \\ &= \mathit{Abst}\, u\, k(x)\, x \\ &= u\, k(x)\, x \\ &= k(x)k(k(x))x \\ &= x\, x, \end{aligned}$$

contradicting Theorem 2.6.

(b) Given u, let $t = Abst\ k(Eq)\langle u, k(p_2)\rangle$ and $s = uk(t)$. Then we have (by Theorems 2.1.d and 2.2a)

$$\begin{aligned} s = u\,k(t) &= Abst\,k(Eq)\langle u, k(p_2)\rangle k(t) \\ &= Eq(\langle u, k(p_2)\rangle k(t)) \\ &= Eq\langle u\,k(t), p_2\rangle \\ &= Eq\langle s, p_2\rangle, \end{aligned}$$

contradicting Theorem 2.4a. □

3. Nonexistence of Various Combinators.

We will show that many standard classical combinators do not exist in TCR. Let us consider the following combinators; none of them is stratified. We use the list presented in [7], with several additions.

$Bxyz = x(yz)$	$Lxy = x(yy)$	$Q_3xyz = z(xy)$
$Cxyz = xzy$	$L_1xy = y(xx)$	$Rxyz = yzx$
$Dxyzw = xy(zw)$	$Mx = xx$	$Sxyz = xz(yz)$
$Fxyz = zyx$	$M_1x = xxx$	$Txy = yx$
$Gxyzw = xw(yz)$	$M_2x = x(xx)$	$Uxy = y(xxy)$
$Hxyz = xyzy$	$Oxy = y(xy)$	$Vxyz = zxy$
$H_1xy = xyx$	$O_1xy = x(yx)$	$Wxy = xyy$
$Jxyzw = xy(xwz)$	$O_2xy = y(yx)$	$W_1xy = yxx$
$Kxy = x$	$Qxyz = y(xz)$	$W_2xy = yxy$
$K_1xy = xx$	$Q_1xyz = x(zy)$	$W_3xy = yyx$

Below we prove that none of these combinators exist in TRC.

(3.1). K_1, K, M and J:

Theorems 2.6, 2.7 and 2.8 show that K_1, K and M do not exist. As for J, it is well known in combinatory logic (cf. [1]) that $\{I, J\}$ is combinatorially complete, and so J cannot exist in TRC.

(3.2). L, O, U and W:

$$M = LI = OI = UI = WI$$

(3.3). O_2 and M_2:

$$M = Abst(O_2 I)I = Abst\ M_2 I :$$
$$Abst(O_2 I)Ix = O_2 I\, k(x)x = k(x)(k(x)I)x = xx$$
$$Abst\ M_2 Ix = M_2 k(x)x = k(x)(k(x)k(x))x = xx$$

(3.4). S and O_1:

$O = SI$ and $S = Abst \circ O_1$ (where $a \circ b$ is the composition, defined in TRC by $a \circ b = Abst\ k(a)(Abst\ k(b)I)$):

$$Sxyz = Abst(O_1 x)yz = O_1 x\, k(z)(yz) = x(k(z)x)(yz) = xz(yz)$$

(3.5). T, C, G, Q_1 and Q_3:

$$K = Abst\ T\, k(I) \text{ and } T = CI = GII = Q_1 I = Q_3 I :$$
$$Kx = Abst\ T\, k(I)x = T\ k(x)(k(I)x) = T\ k(x)I = Ik(x) = k(x)$$

(3.6). B and D:

$$K = AbstBI,\ B = DI :$$
$$Kxy = Abst\ B\ I\, xy = B\, k(x)(Ix)y = B\, k(x)xy =$$
$$= k(x)(xy) = x$$

(3.7). R:

$K = R\, k(I)p_1\langle R, u\rangle R$, where u is arbitrary:

$$\begin{aligned} Kxy &= R\, k(I)p_1\langle R, u\rangle Rxy \\ &= p_1\langle R, u\rangle k(I)R\, xy \\ &= R\, k(I)\, R\, xy \\ &= R\, x\, k(I)y \\ &= k(I)yx \\ &= Ix = x \end{aligned}$$

(3.8). V:

$$K = Abst(V\ Abst)(VI) :$$

using Theorem 2.3c, we have

$$\begin{aligned} Kx &= Abst(V\ Abst)(VI)x \\ &= V\ Abst\ k(x)(VIx) \\ &= VIx\ Abst\ k(x) \\ &= Abst\ Ixk(x) \\ &= k(I)xk(x) \\ &= Ik(x) \\ &= k(x) \end{aligned}$$

(3.9). Q:

$$K_1 = Abst\ Q\ I:$$

$$K_1xy = Abst\ Q\ I\ xy = Q\ k(x)xy = x(k(x)y) = xx$$

(3.10). H_1, H, M_1 and W_2:

$$M_1k(x) = H_1H_1k(x) = W_2W_2k(x) = xk(x),$$

contradicting Theorem 2.9, and $H_1 = HI$.

(3.11). F and W_1:

Let $u = Abst(Fz)Abst$ (where z is arbitrary) and $v = Abst\ W_1\ Abst$. Then $u\,k(x) = v\,k(x) = x\,k(x)$, contradicting Theorem 2.9: using Theorem 2.1d, we have

$$\begin{aligned} uk(x) &= Abst(Fz)Abst\ k(x) \\ &= Fz\ k(k(x))(Abst\ k(x)) \\ &= Abst\ k(x)\ k(k(x))\ z \\ &= x(k(k(x))z) \\ &= x\,k(x) \end{aligned}$$

and

$$\begin{aligned} vk(x) &= Abst\ W_1\ Abst\ k(x) \\ &= W_1\ k(k(x))\,(Abst\ k(x)) \\ &= Abst\ k(x)\ k(k(x))\ k(k(x)) \\ &= x\,(k(k(x))\ k(k(x))) \\ &= x\,k(x). \end{aligned}$$

(3.12). L_1:

Let $a = k(\langle Eq, k(p_2)\rangle)$. Then for all x,

$$\begin{aligned} L_1 a(L_1 x) &= L_1 x(aa) \\ &= aa(xx) \\ &= k(\langle Eq, k(p_2)\rangle)a(xx) \\ &= \langle Eq, k(p_2)\rangle(xx) \\ &= \langle Eq(xx), p_2\rangle, \end{aligned}$$

which, by Theorem 2.4c, is not equal to xx.

Now let $b = Abst\, k(L_1 a)L_1$. By Theorem 2.1d we have

$$bb = Abst\ k(L_1 a)L_1 b = L_1 a(L_1 b),$$

a contradiction.

(3.13). W_3:

Let $a = k(\langle Eq, k(p_2)\rangle)$. Then for all x,

$$\begin{aligned} W_3 x\, a &= a\, a\, x \\ &= \langle Eq, k(p_2)\rangle x \\ &= \langle Eq\ x, p_2\rangle, \end{aligned}$$

which, by Theorem 2.4c, is not equal to x.

Now let $b = Abst\ k(W_3)(W_3 a)$. By Theorem 2.1d we have

$$W_3 ab = bba = Abst\ k(W_3)(W_3 a)ba = W_3(W_3 ab)a.$$

Thus if above we let $x = W_3 ab$, we get

$$W_3(W_3 ab)a \neq W_3 ab,$$

a contradiction.

4. Addendum.

Prof. Holmes brought to my attention a modification of the system TRC that is known to be consistent and to which the results of Section 3 apply as well. The theory TRCL is obtained from TRC by dropping Extensionality (Axiom VII) and adjoining the existence of a constant L such that if $xz = yz$ for all z, then $Lx = Ly$ (as well as several additional axioms about L stated below.) Unlike TRC, the theory TRCL is known to have a model; it is equiconsistent with NFU + Infinity.

The axioms of TRCL are the following: Axioms I-VI and VIII, $Lxy = xy, L(Lx) = Lx, L\langle x,y\rangle = \langle Lx, Ly\rangle, Lp_i = p_i, L\,Abst = Abst, L\,Eq = Eq, Lk(x) = k(x), L(Abst\,x) = Abst\,x, L(Abst\,x\,y) = Abst\,x\,y$, and: if for all z, $xz = yz$, then $Lx = Ly$. The theory TRCL is weaker than TRC, which is equivalent to TRCL + $L = I$.

Theorem 4.1. *The following equalities hold in TRCL:*

$Abst(Abst(Abst\,x)) = Abst\,x, Abst\,k(k(x)) = k(k(x)), Abst\,k(x)yz = x(yz), Abst\,k(x)k(y) = k(xy)$,

$\langle k(x), y\rangle z = \langle x, yz\rangle, \langle x, k(y)\rangle z = \langle xz, y\rangle, k(\langle x,y\rangle) = \langle k(x), k(y)\rangle, k(p_i x) = p_i k(x), p_i(xy) = p_i xy$,

$Abst\langle x,y\rangle = \langle Abst\,x, Abst\,y\rangle, Abst\,p_i = k(p_i), Abst\,I = k(I)$.

Thus all the equalities in Theorems 2.1, 2.2 and 2.3 hold in TRCL as well, with the exception of three formulas that we'll discuss below.

Proof. While TRCL does not have Extensionality, the fact that L fixes abstracts and weak extensionality will do. For example, for 2.1c we argue as follows:

$Abst\,k(k(x))yz = x = k(x)z, L(Abst\,k(k(x))y) = L(k(x), Abst\,k(k(x))y = k(x) = k(k(x))y, L(Abst\,k(k(x))) = Lk(k(x)), Abst\,k(k(x)) = k(k(x))$. □

Theorems 2.4–2.9 are also provable in TRCL, either verbatim or with a slight modification of the proofs in Section 2. One minor difference is in Theorem 2.7: in the proof of Theorem 2.8 we used the following stronger version of 2.7:

Theorem 4.2. *There is no K_1 such that $K_1xy = xx$.*

(This of course follows from 2.7 and Extensionality.)

Proof. Assume that K_1 is such, and let $K_1' = L \circ K_1 = Abst\,k(L)(Abst\,k(K_1)I)$. Since $K_1xy = k(xx)y$, we have $L(K_1x) = Lk(xx) = k(xx)$ and so $K_1'x = (L \circ K_1)x = k(xx)$, contradicting 2.7. □

With these modifications, all proofs in Section 3 go through in TRCL, and we have

Theorem 4.3. *None of the combinators $B, C, ..., W_3$ exist in TRCL.*

As for the remaining equalities in Theorems 2.1 and 2.3, we will show that in TRCL they are all equivalent to Extensionality.

Theorem 4.4. $LL = \mathit{Abst}\, k(I)$.

Proof. Since $\mathit{Abst}\, k(I)xy = xy$, we have $Lx = L(\mathit{Abst}\, k(I)x) = \mathit{Abst}\, k(I)x$, and so $LL = L(\mathit{Abst}\, k(I)) = \mathit{Abst}\, k(I)$. □

Theorem 4.5. *Each of the following equalities implies Extensionality:*
$\mathit{Abst}\, k(I) = I, \mathit{Abst}\, k(p_i) = p_i, \mathit{Abst}(\mathit{Abst}\, k(x)) = k(x)$.

Proof. If $LL = I$ then $Lx = LLx = x$ for all x.

Assume $\mathit{Abst}\, k(p_1) = p_1$. Then for every x, $p_1 x = \mathit{Abst}\, k(p_1)x = L(\mathit{Abst}\, k(p_1)x) = L(p_1 x)$. Moreover, for every x, $p_2 x = p_1\langle p_2, p_1\rangle x = p_1\langle p_2 x, p_1 x\rangle = L(p_1\langle p_2 x, p_1 x\rangle) = L(p_2 x)$. Thus $Lx = L\langle p_1 x, p_2 x\rangle = \langle L(p_1 x), L(p_2 x)\rangle = \langle p_1 x, p_2 x\rangle = x$, for all x.

If $\mathit{Abst}(\mathit{Abst}\, k(x)) = k(x)$ then (for any y) $Lx = L(k(x)y) = L(\mathit{Abst}(\mathit{Abst}\, k(x))y) = \mathit{Abst}(\mathit{Abst}\, k(x))y = k(x)y = x$. □

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