

8. THE RADON-NIKODYM PROPERTY

8.1. INTRODUCTION.

8.1.1. A Banach space X is said to have the Radon-Nikodym Property (RNP) provided for every σ -algebra Σ , for every μ in $ca^+(\Sigma)$, and for every v in $bv(\Sigma, X, \mu)$, v has a Bochner integrable derivative with respect to μ .

For example, we already know from 7.3.5. that every reflexive space has the RNP. From 7.1.2, $C[0,1]$ does not have the RNP. Section 8.2 gives these and other examples of spaces with and without the RNP. Section 8.3 is devoted to a geometric characterization of spaces with the RNP.

This section is devoted to results which reduce the requirements for the RNP. We start by formulating the proof of 7.3.4.

8.1.2. THEOREM. If X is a Banach space, then X has the RNP if and only if for every σ -algebra Σ , for every μ in $ca^+(\Sigma)$, and for every v in $bv(\Sigma, X, \mu)$ with bounded average range, v has a Bochner integrable derivative with respect to μ .

PROOF. Suppose the condition holds and let v be any member of $bv(\Sigma, X, \mu)$. By the Radon-Nikodym Theorem (3.1.9.), choose $h \geq 0$ in $L^1(\mu)$ such that

$|v|(E) = \int_E h d\mu$ ($E \in \Sigma$). For each $n = 1, 2, \dots$ let $E_n = \{\omega : h(\omega) \leq n\}$, and let $v_n(E) = v(E \cap E_n)$. Then

$$\|v_n(E)\| \leq \int_{E \cap E_n} h d\mu \leq n\mu(E), \text{ so } v_n \text{ has bounded average}$$

range. For each n choose g_n such that $v_n(E) = \int_E g_n d\mu$ ($E \in \Sigma$). Note that $E_n \uparrow \Omega$ and we may assume that $m < n \Rightarrow g_n = g_m$ on E_m . Hence, $g(\omega) = \lim g_n(\omega)$ exists for all ω . Clearly g is strongly measurable. Also, by the monotone convergence theorem $\int \|g\| d\mu = \lim \int \|g_n\| d\mu = \lim \|v_n\|(\Omega) = \lim \int_E h d\mu = \int h d\mu = |v|(\Omega) < \infty$.

Thus g is Bochner integrable. By countable additivity, for any E in Σ

$$v(E) = \lim v_n(E) = \lim \int_E g_n d\mu = \lim \int_{E \cap E_n} g d\mu = \int_E g d\mu.$$

8.1.3. We use the notation $bv_0(\Sigma, X, \mu)$ for the space of all v in $bv(\Sigma, X, \mu)$ with bounded average range.

8.1.4. LEMMA. If $\mu \in ca^+(\Sigma)$ is purely atomic, then for any Banach space X , every v in $bv(\Sigma, X, \mu)$ has a Bochner integrable derivative with respect to μ .

PROOF. Let $\Omega = \bigcup_{n=1}^{\infty} A_n$ where each A_n is an atom of μ . Define g by

$$g = \sum_{n=1}^{\infty} \frac{v(A_n)}{\mu(A_n)} X_{A_n}.$$

Then g is in $L^1(\mu, X)$ and $\int_A g d\mu = v(A) , \forall A \in \Sigma$.

8.1.5. LEMMA. A Banach space X has the RNP if and only if for every countably generated σ -algebra Σ , every v in $ca^+(\Sigma)$, and every v in $bv_0(\Sigma, X, \mu)$, v has a Bochner integrable derivative with respect to μ .

PROOF. This follows from 7.5.2.

8.1.6. THEOREM. Let M denote the σ -algebra of Borel subsets of $[0,1]$, and let m denote Lebesgue measure on M . Then a Banach space X has the RNP if and only if every v in $bv_0(M, X, m)$ has a Bochner integrable derivative with respect to m .

PROOF. Assume the condition holds. Let Σ be a countably generated σ -algebra, let μ be in $ca^+(\Sigma)$, and let v be in $bv_0(\Sigma, X, \mu)$. Let Ω be expressed as a disjoint union of $A, B \in \Sigma$ as in 2.2.8 (i.e., A is essentially the union of all atoms of μ), let $\mu_1(E) = \mu(E \cap A)$, $\mu_2(E) =$

$\mu(E \cap B)$, $v_1(E) = v(E \cap A)$, and $v_2(E) = v(E \cap B)$ for all E in Σ .

Since μ_1 is purely atomic and v_1 is in $bv_0(\Sigma, \mathbb{X}, \mu_1)$, there exists g in $L^1(\mu_1, \mathbb{X})$ with $v_1(E) = \int_E g d\mu_1$ for all E in Σ . Clearly we may assume g is supported on A .

If $\mu_2 = 0$, we are done. If $\mu_2 \neq 0$, we may assume $\mu_2(\Omega) = 1$. Since μ_2 is non-atomic and Σ is countably generated, by 2.2.10, there exists an isomorphism φ mapping Σ/μ_2 onto \mathbb{N}/m . Define $\sigma: M \rightarrow \mathbb{X}$ by

$$\sigma(I) = \bar{v}_2(\varphi^{-1}(I)) , \quad (I \in \mathbb{N}).$$

Then σ is in $bv_0(M, \mathbb{X}, \mu)$, so by hypothesis there exists f in $L^1(\mathbb{N}, \mathbb{X})$ such that

$$\sigma(I) = \int_I f dm , \quad (I \in \mathbb{N}).$$

For every finite Σ -partition $\pi = (E_1, \dots, E_n)$ of Ω , define $hw \in L^1(\mu_2, \mathbb{X})$ and $fw \in L^1(\mathbb{N}, \mathbb{X})$ by

$$hw = \sum_{i=1}^n \frac{v_2(E_i)}{\mu_2(E_i)} \chi_{E_i}$$

and

$$f_{\pi} = \sum_{i=1}^n \frac{v_2(E_i)}{\mu_2(E_i)} x_{\phi(\bar{E}_i)}$$

$$= \sum_{i=1}^n \frac{\overline{o}(\phi(\bar{E}_i))}{\overline{m}(\phi(\bar{E}_i))} x_{\phi(\bar{E}_i)},$$

where $x_{\phi(\bar{E}_i)}$ has its obvious meaning, being defined almost everywhere. It follows from 7.1.6 that

$$\| f_{\pi} - f \|_1 \rightarrow 0. \text{ Now for } \pi = E_1, \dots, E_n \text{ and } \pi^1 = (F_1, \dots, F_m)$$

we have

$$\begin{aligned} \| h_{\pi} - h_{\pi^1} \|_1 &= \left\| \sum_{i=1}^n \frac{v_2(E_i)}{\mu_2(E_i)} x_{E_i} - \sum_{j=1}^m \frac{v_2(F_j)}{\mu_2(F_j)} x_j \right\|_1 \\ &= \sum_{i,j} \left\| \frac{v_2(E_i)}{\mu_2(E_i)} - \frac{v_2(F_j)}{\mu_2(F_j)} \right\| \mu(E_i \cap F_j) \\ &= \sum_{i,j} \left\| \frac{v_2(E_i)}{\mu_2(E_i)} - \frac{v_2(F_j)}{\mu_2(F_j)} \right\| \overline{m}(\phi(\bar{E}_i) \cap \phi(\bar{F}_j)) \\ &= \| f_{\pi} - f_{\pi^1} \|_1 \xrightarrow{\pi \rightarrow \pi^1} 0. \end{aligned}$$

Thus there exists h in $L^1(\mu_2, X)$ such that $\| h_{\pi} - h \|_1 \rightarrow 0$, and it follows that $\int_E h d\mu_2 = v_2(E)$, all $E \in \Sigma$. Clearly we can assume h is supported on B .

Finally, $g + h$ is in $L^2(\mu, X)$ and for all
 E in Σ ,

$$\begin{aligned} \int_E (g + h) d\mu &= \int_E g d\mu_1 + \int_E h d\mu_2 \\ &= v_1(E) + v_2(E) \\ &= v(E). \end{aligned}$$

8.1.7. COROLLARY. For a Banach space X , the following three statements are equivalent.

- (1) X has the RNP.
- (2) Every closed subspace of X has the RNP.
- (3) Every separable closed subspace of X has the RNP.

PROOF. (1) \Rightarrow (2) by 7.1.9, (2) \Rightarrow (3) clearly, and (3) \Rightarrow (1)^(*)
by 8.1.7 (or 8.1.6) since every v in $bv(M, X, m)$ takes its values in a separable subspace of X .

8.2. CLASSICAL SPACES WITH AND WITHOUT THE RNP

8.2.1. A Banach space X is said to be a dual space if it is isomorphic to Y^* for some Banach space Y .

A Banach space X is said to be weakly compactly generated (WCG) if and only if there is a weakly compact set K in X such that K is the closed linear span of K . Clearly every reflexive space is WCG. Also any separable space is compactly generated (viz., if (x_n) is a dense sequence in X consider the set

$$K = \{0\} \cup \left(\frac{x_n}{n\|x_n\|} \right).$$

We show that a WCG dual space has the RNP. The following fundamental lemma is a special case.

8.2.2. THEOREM (Dunford and Pettis). A separable dual space X has the RNP.

PROOF. Assume $X = Y^*$ and X is separable. Let v be in $bv(\Sigma, X, \mu)$, where μ is in $ca^+(\Sigma)$, and assume v has bounded average range. We use 8.1.2. For every finite Σ -partition $\pi = (E_1, \dots, E_n)$ of Ω , let

$$e_\pi = \sum_{i=1}^n \frac{v(E_i)}{\mu(E_i)} x_{E_i}$$

Then each g_{π} takes its range in the average range of v which is conditionally weak* compact. Hence there is a subset $(g_{\pi})_1$ of $(g_{\pi})_N$ such that $g_{\pi}1 \rightarrow g$ pointwise weak* for some g mapping Ω into the weak* closure of the average range of v . For every y in Σ , $\hat{y} \circ v$ is in $ca(\Sigma, \mu)$ and hence there exists h_y in $L^1(\mu)$ such that

$$\hat{y} \circ v(E) = \int_E h_y d\mu, \quad \forall E \in \Sigma.$$

Since $\hat{y} \circ v$ has bounded average range, h_y is in $L^\infty(\mu)$ by 7.1.4. If E_{π} is as defined in 7.1.6, then

$$\begin{aligned} E_{\pi}(h_y) &= \sum_n \frac{1}{\mu(E_n)} \left(\int_{E_n} h_y d\mu \right) \chi_{E_n} \\ &= \sum_n \frac{\hat{y}(v(E_n))}{\mu(E_n)} \chi_{E_n} \\ &= \hat{y} \circ \left(\sum_n \frac{v(E_n)}{\mu(E_n)} \chi_{E_n} \right) \\ &= \hat{y} \circ g_{\pi}. \end{aligned}$$

By 7.1.6, $E_{\pi}(h_y) \rightarrow h_y$ in $L^\infty(\mu)$, and by the choice of g we must have $\hat{y} \circ g_1 \rightarrow \hat{y} \circ g$ pointwise. Hence $h_y = \hat{y} \circ g$. In particular, $\hat{y} \circ g$ is measurable for every

y in Y . Let x^* be in $X^* = Y^{**}$, and using
and , choose a sequence (y_n) in Y with
 $\|y_n\| \leq \|x^*\|$ such that $x^*(x) = \lim \hat{y}_n(x)$ for every
 x in X . Then $x^* \circ g = \lim \hat{y}_n \circ g$ pointwise, and hence
 $x^* \circ g$ is measurable. Thus g is weakly measurable and
thus strongly measurable by 6.2.5. Since g is bounded
it is Bochner integrable. Finally, if A is in Σ
and y is in Y , then

$$\left(\int_A g d\mu \right) (y) = \int_A \hat{y} \circ g d\mu = \int_A y d\mu = v(E)(y).$$

8.2.3. COROLLARY (uhl). If Y is a space such that every
separable subspace of Y has a separable dual, then
 Y^* has the RNP.

PROOF. We use 8.1.7. Let Z be a separable closed sub-
space of Y^* , say Z is the closure of (y_n^*) . For
each n , choose a sequence $(y_m^{(n)})_{m=1}^\infty$ in the unit ball
of Y such that $y_n^* (y_m^{(n)}) \xrightarrow{m} \|y_n^*\|$. If $y_n^* \rightarrow y^*$ in Y^* ,
then

$$\begin{aligned} |y^*(y_m^{(n)}) - \|y^*\|| &\leq |y^*(y_m^{(n)}) - y_n^*(y_m^{(n)})| + |y_n^*(y_m^{(n)}) - \|y_n^*\|| \\ &\quad + |\|y_n^*\| - \|y^*\||, \end{aligned}$$

and this is small for m and n large. Let W denote the closed linear span of $\{y_m^{(n)} : m, n = 1, 2, \dots\}$.

Then by hypotheses Y^* is separable. By the above computation, the restriction map $R: Z \rightarrow W^*$, where $R(y^*) = y^*|_W$, is an isometry. Hence by 8.2.2 and 8.1.7, Z has the RNP. By 8.1.7 again, Y^* has the RNP.

8.2.4. REMARK. C. Stegall has announced that the converse of 8.2.3 holds. Thus Y^* has the RNP if and only if every separable subspace of Y has a separable dual. Stegall's proof is not yet available.

8.2.5. LEMMA. If Y is a separable space and if Y^* is WCG, then Y^* is separable.

PROOF. Let K be a weakly compact subset of Y^* . Then $(K, \text{weak}) = (K, \text{weak}^*)$. By (K, weak^*) is metrizable, so (K, weak) is a compact metric space and hence separable.

8.2.6. THEOREM. If X is a WCG dual space, then X has the RNP.

PROOF. Let $X = Y^*$ and let W be a separable subspace of Y . By 8.2.3 we need only show W^* is separable. But W^* is a quotient space of X and hence is WCG, so by 8.2.5, W^* is separable.

8.2.7. COROLLARY (Phillips). Every reflexive space has the RNP.

REMARK. We of course already knew this by 7.3.5. The result is stated here to emphasize that it is covered by 8.2.6.

8.2.8. COROLLARY. For any set Ω , $\ell^1(\Omega)$ has the RNP.

PROOF. If Ω is countable, then $\ell^1(\Omega)$ is a separable dual space (the dual of c_0), and so it has the RNP. In the general case we use 8.1.7. Let X be a separable subspace of $\ell^1(\Omega)$ and let $(x_n)_{n=1}^\infty$ be a dense sequence in X . Let Q denote the set of all ω in Ω for which $x_n(\omega) \neq 0$ for some n . Then the restriction map $R: X \rightarrow \ell^1(Q)$, given by $R(x) = x|Q$, is an isometry. By 8.1.7, and the fact that Q is countable, X has the RNP. Again by 8.1.7, $\ell^1(\Omega)$ has the RNP.

8.2.9. REMARK. For uncountable Ω , $\ell^1(\Omega)$ is a dual space which has the RNP but it is not WCG (Hence the converse of 8.2.6 fails). To see that $\ell^1(\Omega)$ is not WCG, suppose it is and let K be a weakly compact set in $\ell^1(\Omega)$ with closed linear span all of $\ell^1(\Omega)$. By 5.2.2 and the Eberlein-Smulian

Theorem, K is compact and hence separable. This implies $\ell^1(\Omega)$ is separable which is impossible if Ω is uncountable.

8.2.10. THEOREM. The space c_0 does not have the RNP. Thus no Banach space which contains a copy of c_0 has the RNP.

PROOF. Let M denote the σ -algebra of Borel subsets of $[0,1]$, and let m denote Lebesgue measure on M . For each $i = 1, 2, \dots$, choose n such that $2^n \leq i < 2^{n+1}$, and let

$$A_i = \left[\frac{i - 2^n}{2^n}, \frac{i - 2^n + 1}{2^n} \right].$$

Define $v: M \rightarrow c_0$ by $(v(A))_i = m(A \cap A_i)$. Then v is finitely additive and $\|v(A)\|_\infty \leq m(A)$ for all A in M . Hence v is in $bv(M, c_0, m)$.

Suppose $f: [0,1] \rightarrow c_0$ is a derivative of v with respect to m . For every A in M and any i ,

$$\int_A (f(s))_i dm(s) = (v(A))_i = m(A \cap A_i),$$

so there exists an m -null set $C_i \in M$ such that

$(f(s))_i = \chi_{A_i}(s)$ for all $s \in [0,1] \setminus C_i$. Let γ be in

$[0,1] \setminus \bigcup_{i=1}^{\infty} C_i$. Then $\chi_{A_i}(s) = 1$ for infinitely many i ,

so $\lim_{i \rightarrow \infty} (f(s))_i \neq 0$. This contradicts the assumption that $f(s)$ is in c_0 .

8.2.11. THEOREM. If $\lambda \in ca^+(\Sigma)$ is not purely atomic, then $L^1(\lambda)$ does not have the RNP. Thus no Banach space which contains a copy of $L^1(\lambda)$ has the RNP.

PROOF. Suppose $\lambda \in ca^+(\Sigma)$ is not purely atomic, and let $v : \Sigma \rightarrow L^1(\lambda)$ be given by $v(E) = x_E$. Then v is finitely additive and $\|v(E)\|_1 = \lambda(E)$, so v is in $bv(\Sigma, L^1(\lambda), \lambda)$. Suppose v has a derivative g with respect to λ . Since v has bounded average range, g is in $L^\infty(\lambda, L^1(\lambda))$ by 7.1.4, and hence by 7.3.1, v has conditionally compact range. Choose $(E_n) \subset \Sigma$ as in 2.2.12. Then for $n \neq m$,

$$\|v(E_n) - v(E_m)\|_1 = \|x_{E_n} - x_{E_m}\|_1 = \lambda(E_n \Delta E_m) \geq \epsilon,$$

so $(v(E_n))_{n=1}^\infty$ cannot have a convergent subsequence, a contradiction.

8.3 A GEOMETRIC CHARACTERIZATION OF SPACES WITH THE RNP

In 7.4.5 it was observed that if v is in $bv(\Sigma, X, \lambda)$ and if every bounded subset of $AR(v, \Omega; \lambda)$ is dentable, then v has a derivative in $L^1(\lambda, X)$. It follows that if X is any Banach space such that every bounded subset is dentable, then X has the RNP. In this section we show that the converse also holds.

Although the following lemma is a consequence of 7.4.4, the proof there is not direct. We give an elementary direct proof here.

8.3.1. LEMMA. Let λ be in $ca^+(\Sigma)$, $\lambda \neq 0$, let g be in $L^1(\lambda, X)$, and let $v(E) = \int_E g d\lambda$, ($E \in \Sigma$). Then for every $\delta > 0$ there exists E in Σ with $\lambda(E) > 0$ and $\text{diam}(AR(v, E; \lambda)) \leq \delta$.

PROOF. Choose simple functions g_n such that $g_n \rightarrow g$ a.e.

For $k = 1, 2, \dots$, let

$$E_k = \{\omega : \|g_n(\omega) - g(\omega)\| \leq \frac{\delta}{2}, n \geq k\}.$$

Then $\lambda(\bigcup_{k=1}^{\infty} E_k) = \lambda(\Omega)$, so there exists k with $\lambda(E_k) > 0$.

Let $s_k = \sum_{i=1}^n x_i \chi_{A_i}$, where the A_i 's form a partition

of Ω , and choose any i for which $\lambda(A_i \cap E_k) > 0$. Let $E = A_i \cap E_k$. Then for any $F \subset E$ with $\lambda(F) > 0$,

$$\begin{aligned} \|v(F) - x_i \lambda(F)\| &= \left\| \int_F g d\lambda - \int_F x_i d\lambda \right\| \\ &\leq \int_F \|g - x_i\| d\lambda \\ &\leq \frac{\delta}{2} \lambda(F), \end{aligned}$$

so $\left\| \frac{v(F)}{\lambda(F)} - x_i \right\| \leq \frac{\delta}{2}$. Hence $\text{diam } (\text{AR}(v, E; \lambda)) \leq \delta$.

6.3.2. THEOREM. A Banach space X has the RNP if and only if every bounded subset of X is dentable.

PROOF. (\Leftarrow). This follows from 7.4.4 and 8.1.2 (or by 7.4.5).

(\Rightarrow). Let K be a bounded set in X (with bound $M \geq 1$), and assume K is not dentable. Let $\epsilon > 0$ be such that

$$(*) \quad x \in K \rightarrow x \in \overline{\text{co}}(K \setminus B_\epsilon(x)).$$

We may assume $\epsilon < 1$.

Let $\Omega = [0,1]$, and let λ denote Lebesgue measure on $[0,1]$.

We will define by induction an increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \dots$ of finite algebras of subsets of Ω , and additive functions $v_n : \Sigma_n \rightarrow X$ such that the following four statements hold.

(1) The atoms of Σ_n form a partition $(I_1^n, \dots, I_{p_n}^n)$

of Ω into half-open intervals.

(2) $\frac{v_n(I_i^n)}{\lambda(I_i^n)}$ is in K for all n and all $i = 1, \dots, p_n$.

(3) $I_j^{n+1} \subset I_j^n \Rightarrow \left\| \frac{v_{n+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{v_n(I_j^n)}{\lambda(I_j^n)} \right\| \geq \frac{2^n - 1}{2^n} \epsilon$.

(4) $\|v_n(E) - v_{n+1}(E)\| \leq \frac{\epsilon}{2^n} \lambda(E), \quad \forall E \in \Sigma_n,$

Before proceeding, note that (2) and (4) respectively imply

(5) $\|v_n(E)\| \leq M\lambda(E), \quad \forall E \in \Sigma_n,$

and (6) $\left\| \frac{v_n(E)}{\lambda(E)} - \frac{v_{n+1}(E)}{\lambda(E)} \right\| \leq \frac{\epsilon}{2^n}, \quad \forall E \in \Sigma_n, \lambda(E) \neq 0.$

To start the induction, let $\Sigma_0 = (\emptyset, \Omega)$, let x_0 be any element of K , and let $v_0(\emptyset) = 0, v_0(\Omega) = x_0$.

Next, suppose Σ_n and v_n have been defined, $n \geq 0$. For each $i = 1, \dots, p_n$, using (2) and (*), choose finite sets

$(x_1^{(i)}, \dots, x_{q_i}^{(i)}) \subset K$ and $(a_1^{(i)}, \dots, a_{q_i}^{(i)}) \subset (0,1)$ such that $\sum_{j=1}^{q_i} a_j^{(i)} = 1$,

$$\left\| x_j^{(i)} - \frac{v_n(I_i^n)}{\lambda(I_i^n)} \right\| \geq \epsilon, \quad j=1, \dots, q_i,$$

and

$$\left\| \frac{v_n(I_i^n)}{\lambda(I_i^n)} - \sum_{j=1}^{q_i} a_j^{(i)} x_j^{(i)} \right\| < \frac{\epsilon}{2^n}.$$

Partition I_i^n into disjoint half-open intervals $J_1^{(i)}, \dots, J_{q_i}^{(i)}$

such that

$$\lambda(J_j^{(i)}) = \lambda(I_i^n) a_j^{(i)}, \quad \forall j = 1, \dots, q_i.$$

Now let Σ_{n+1} be the algebra generated by $\{J_j^{(i)} : i = 1, \dots, p_n; j = 1, \dots, q_i\}$, and let $v_{n+1} : \Sigma_{n+1} \rightarrow X$ be the additive function satisfying

$$v_{n+1}(J_j^{(i)}) = \lambda(J_j^{(i)}) x_j^{(i)}$$

$$= \lambda(I_i^n) a_j^{(i)} x_j^{(i)}$$

for all $i = 1, \dots, p_n; j = 1, \dots, q_i$.

Then (1) and (2) are clear for the pair Σ_{n+1}, v_{n+1} .

To check (3), note that

$$\begin{aligned} & \left\| \frac{v_{n+1}(J_j^{(i)})}{\lambda(J_j^{(i)})} - \frac{v_{n+1}(I_i^n)}{\lambda(I_i^n)} \right\| = \left\| x_j^{(i)} - \sum_{k=1}^{q_i} \frac{v_{n+1}(J_k^{(i)})}{\lambda(I_i^n)} \right\| \\ &= \left\| x_j^{(i)} - \sum_{k=1}^{q_i} a_k^{(i)} x_k^{(i)} \right\| \\ &\geq \left\| x_j^{(i)} - \frac{v_n(I_i^n)}{\lambda(I_i^n)} \right\| + \left\| \sum_{k=1}^{q_i} a_k^{(i)} x_k^{(i)} - \frac{v_n(I_i^n)}{\lambda(I_i^n)} \right\| \\ &\geq \epsilon - \frac{\epsilon}{2^n} = \frac{2^n - 1}{2^n} \epsilon. \end{aligned}$$

To check (4), note that

$$\begin{aligned}
 \|v_n(I_i^n) - v_{n+1}(I_i^n)\| &= \|v_n(I_i^n) - \sum_{j=1}^{q_i} v_{n+1}(J_j^{(i)})\| \\
 &= \|v_n(I_i^n) - \sum_{j=1}^{q_i} \lambda(I_i^n) \alpha_j^{(i)} x_j^{(i)}\| \\
 &= \lambda(I_i^n) \left\| \frac{v_n(I_i^n)}{\lambda(I_i^n)} - \sum_{j=1}^{q_i} \alpha_j^{(i)} x_j^{(i)} \right\| \\
 &\leq \lambda(I_i^n) \frac{\epsilon}{2^n}.
 \end{aligned}$$

Then (4) follows from the fact that every E in Σ_n is the disjoint union of the I_i^n 's.

Therefore, by induction we have the sequences (Σ_n) , (v_n) such that (1) - (6) are satisfied.

Let $G = \bigcup_{n=0}^{\infty} \Sigma_n$. Then G is an algebra. If E is in G then there exists N such that

$$n \geq N \Rightarrow E \in \Sigma_N.$$

By (4), $v(E) = \lim_n v_n(E)$ exists. Trivially the function $v: G \rightarrow X$ so defined is finitely additive.

By (5),

$$(5') \quad \|v(E)\| \leq M\lambda(E), \quad \forall E \in G,$$

so v is countably additive and s -additive on G . By 5.2.6, v extends to a countably additive measure \bar{v} on $\Sigma = \Sigma(G)$. For any x^* in X^* with $\|x^*\| \leq 1$ we have for any E in Σ

$$\begin{aligned}
 x^* \circ \bar{v}(E) &= \lim_n \sum_i x^* \circ v(A_i) \\
 &\leq M \lim_n \sum_i \lambda(A_i) \quad (\text{by (5)}) \\
 &= M\lambda(E),
 \end{aligned}$$

where the limit is taken over all countable disjoint coverings

$\pi = (A_1, A_2, \dots)$ of E by members of G .

Hence,

$$(7) \quad \|\bar{v}(E)\| \leq M\lambda(E), \quad \forall E \in \Sigma,$$

so \bar{v} is of bounded variation and $\bar{v} \ll \lambda$. If X has the RNP, v has a derivative g in $L^2(\lambda|\Sigma, X)$.

Next we observe

$$(8) \quad n \geq 4, I_j^{n+1} \subset I_i^n \Rightarrow \left\| \frac{v(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{\bar{v}(I_i^n)}{\lambda(I_i^n)} \right\| \geq \frac{\epsilon}{2}.$$

For if $n \geq 4$, then for any $m \geq n + 1$,

$$\begin{aligned}
 &\left\| \frac{v_m(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{v_m(I_i^n)}{\lambda(I_i^n)} \right\| \\
 &\geq \left\| \frac{v_{n+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{v_{n+1}(I_i^n)}{\lambda(I_i^n)} \right\| + \left(\left\| \frac{v_m(I_i^n)}{\lambda(I_i^n)} - \frac{v_{n+1}(I_i^n)}{\lambda(I_i^n)} \right\| \right. \\
 &\quad \left. + \left\| \frac{v_{n+1}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{v_m(I_j^{n+1})}{\lambda(I_j^{n+1})} \right\| \right)
 \end{aligned}$$

$$\geq \frac{2^n - 1}{2^n} \epsilon - \frac{4\epsilon}{2^n} \quad (\text{using (3) and (6)})$$

$$= \frac{2^n - 5}{2^n} \epsilon > \frac{\epsilon}{2}.$$

Taking the limit as $n \rightarrow \infty$ gives (8).

We shall now get a contradiction to the lemma. Let B be in Σ with $\lambda(B) > 0$. We show

$$\text{diam } (\text{AR}(v, B; \lambda|\Sigma)) \geq \frac{\epsilon}{4},$$

which will contradict the lemma and complete the proof.

Since $\Sigma = \Sigma(G)$, there exists E in G such that

$$\lambda(E \setminus B) + \lambda(B \setminus E) = \lambda(B \Delta E) < \frac{\epsilon}{16M} \lambda(B).$$

$$\begin{aligned} \text{Then } \lambda(E \setminus B) &< \frac{\epsilon}{16M} \lambda(B) - \lambda(B \setminus E) \\ &\leq \frac{\epsilon}{16M} [\lambda(B) - \lambda(B \setminus E)] \\ &= \frac{\epsilon}{16M} \lambda(B \cap E). \end{aligned}$$

Now since E is in G , we can choose $n \geq 4$ such that E is in Σ_n . Then E is a (disjoint) union of a subcollection of $(I_1^n, \dots, I_{p_n}^n)$. We claim that there is some i such that

$$\lambda(I_i^n \setminus B) < \frac{\epsilon}{16M} \lambda(I_i^n \cap B).$$

For if not, then

$$\lambda(E \setminus B) = \sum_{i, I_i^n \subset E} \lambda(I_i^n \setminus B)$$

$$\geq \frac{\epsilon}{16M} \sum_{i, I_i^n \subset E} \lambda(I_i^n \cap B)$$

$$= \frac{\epsilon}{16M} \lambda(E \cap B),$$

contradicting the choice of E . Thus the claim must hold.

Let $C = I_1^n \cap B$. Then $C \subset B$ and

$$(4) \quad \lambda(I_1^n \setminus C) < \frac{\epsilon}{16M} \lambda(C).$$

In particular $\lambda(C) > 0$.

As above, we claim that there exists $I_j^{n+1} \subset I_1^n$
such that

$$\lambda(I_j^{n+1} \setminus C) < \frac{\epsilon}{16M} \lambda(I_j^{n+1} \cap C).$$

For if not, then

$$\lambda(I_1^n \setminus C) = \sum_{j, I_j^{n+1} \subset I_1^n} \lambda(I_j^{n+1} \setminus C)$$

$$\geq \frac{\epsilon}{16M} \sum_{j, I_j^{n+1} \subset I_1^n} \lambda(I_j^{n+1} \cap C)$$

$$= \frac{\epsilon}{16M} \lambda(I_1^n \cap C)$$

$$= \frac{\epsilon}{16M} \lambda(C),$$

contradicting (*). Thus the claim holds. Let

$$D = I_j^{n+1} \cap C. \text{ Then } D \subset B \text{ and}$$

$$\lambda(I_j^{n+1} \setminus D) < \frac{\epsilon}{16M} \lambda(D),$$

and in particular $\lambda(D) > 0$.

By (8),

$$\left\| \frac{\bar{v}(I_1^{n+1})}{\lambda(I_j^{n+1})} - \frac{\bar{v}(I_1^n)}{\lambda(I_1^n)} \right\| \geq \frac{\epsilon}{2}.$$

But also,

$$\begin{aligned} \left\| \frac{\bar{v}(I_1^n)}{\lambda(I_1^n)} - \frac{\bar{v}(C)}{\lambda(C)} \right\| &= \left\| \frac{\lambda(I_1^n \setminus C)}{\lambda(C)} \left[\frac{\bar{v}(I_1^n \setminus C)}{\lambda(I_1^n \setminus C)} - \frac{\bar{v}(I_1^n)}{\lambda(I_1^n)} \right] \right\| \\ &\leq \frac{\lambda(I_1^n \setminus C)}{\lambda(C)} \left(\left\| \frac{\bar{v}(I_1^n \setminus C)}{\lambda(I_1^n \setminus C)} \right\| + \left\| \frac{\bar{v}(I_1^n)}{\lambda(I_1^n)} \right\| \right) \\ &< \frac{\epsilon}{16M} \cdot 2M \quad (\text{by (*) and (7)}), \\ &= \frac{\epsilon}{8}, \end{aligned}$$

and similarly

$$\left\| \frac{\bar{v}(I_j^{n+1})}{\lambda(I_j^{n+1})} - \frac{\bar{v}(D)}{\lambda(D)} \right\| < \frac{\epsilon}{8}.$$

Thus

$$\left\| \frac{v(C)}{\lambda(C)} - \frac{\tilde{v}(D)}{\lambda(D)} \right\| \geq \frac{\epsilon}{4},$$

so

$$\text{diam } (\text{AR}(\bar{v}, B; \lambda|\Sigma)) \geq \frac{\epsilon}{4}.$$

8.3.3. It should be observed that by forcing $\lambda(I_1^n) \leq \frac{1}{n}$ for every n in the above proof, one obtains Σ equal to the Borel subsets of $[0,1)$. Thus 8.1.6 is a consequence of the above construction.

R.8.1. REMARKS AND REFERENCES.

Theorem 8.1.6 as proved here was given by Chatterji [1968]. This result also follows from the construction in the proof of 8.3.2 below. Uhl [1972-73] also gives the following proof of the non-triviality (3) \Rightarrow (1) of Corollary 8.1.7 which is independent of the above results.

(3) \Rightarrow (1) in 8.3.2. Let v be in $bv_0(\Sigma, X, \mu)$. We need only show v has separable range. We show in fact that v must have conditionally compact range if (3) holds. Let $\{E_n\}$ be any sequence in Σ , and let Σ_0 be the σ -algebra generated by $\{E_n\}$. We have that $v(\Sigma_0)$ is separable, and hence $v|\Sigma_0$ has a derivative g which is in $L^\infty(\mu|\Sigma_0, X)$ by 7.1.4. Thus by 7.3.1, $v|\Sigma_0$ had conditionally norm compact range, so some subsequence of $\{v(E_n)\}$ converges in norm.

Additional References: Diestel and Faires [1973].

R.8.2. REMARKS AND REFERENCES.

(1) Of the classical sequence spaces ℓ^p has the RNP for $1 \leq p < \infty$, while c_0, c , and $\ell^\infty = m$ do not. For λ not purely atomic, $L^1(\lambda)$ and $L^\infty(\lambda)$ do not have the RNP while of course all $L^p(\lambda)$ for $1 < p < \infty$ do. For an infinite compact Hausdorff space Ω , $C(\Omega)$ does not have the RNP.

(2) As pointed out by Uhl [1972], the quasi-reflexive spaces introduced by James [1950] and named and studied by Civin and Yood [1957], have the RNP. (Civin and Yood [1957] showed that every subspace of a quasi-reflexive space is quasi-reflexive, and that every quasi-reflexive space is a dual space. Hence every separable subspace is a separable dual space.)

(3) In this section, the fundamental theorem 8.2.2 is due to Dunford and Pettis [1904], 8.2.3 was observed by Uhl [1972], and 8.2.6 was noted by W.B. Johnson, the proof here being due to C. Stegall. The remark 8.2.9 is due to Diestel [1973]. Theorem 8.2.7 of course is due originally to Phillips [1943]. The example used here to prove Theorem 8.2.10 is from Lewis [1972].

(4) Preprints of the paper by Stegall mentioned in 8.2.4 are now available (Stegall [1973]).

Additional References: Diestel and Faires[1973].

R.8.3. REMARKS AND REFERENCES.

(1) In order to discuss the history of Theorem 8.3.2 we need to introduce the notion of σ -dentable if and only if for every $\epsilon > 0$ there exists some x in K such that

$$x \notin \sigma\text{-co}(K \cap B_\epsilon(x)),$$

where $\sigma\text{-co}(K \cap B_\epsilon(x)) = \{\sum_{i=1}^n \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1, x_i \in K \cap B_\epsilon(x)\}$.

Maynard [1973] noted that Rieffel's theorem 7.4.4 and its proof remain valid if dentability is replaced everywhere by σ -dentability. Conversely, he showed that if X contains a bounded set which is not σ -dentable, then X does not have the RNP; hence X has the RNP if and only if every bounded set is σ -dentable. By modifying Maynard's construction, Huff [1973] proved the stronger result, Theorem 8.3.2, the proof being that given here. At the same time, Davis and Phelps [1973] showed that if X has a bounded non-dentable subset, then it has a bounded non- σ -dentable subset.

It should be pointed out that Maynard [1973] gives an example of a bounded set which is σ -dentable but not dentable. (All such sets must of course lie in a space without the RNP.)

Additional References: Diestel and Faires [1973].