6.1 INTEGRALS OF WEAKLY MEASURABLE FUNCTIONS

Throughout this section, Σ is a σ-algebra of subsets of Ω and μ is a fixed member of $ca^+(\Sigma)$. We shall develop an integration theory for functions $f:\Omega \to X$, where X is a Banach space. For convenience we assume $A \subset B \subset \Omega$, $B \in \Sigma$ $\mu(B) = 0 \Rightarrow A \in \Sigma$, i.e., μ is complete.

6.1.1 A function $f:\Omega \to X$ is said to be <u>weakly</u> (Σ-)<u>measurable</u> provided $x^* \circ f$ is Σ-measurable for every x^* in X^*. A weakly measurable function f is said to be <u>integrable</u> provided $x^* \circ f$ is in $L^1(\mu)$ for every x^* in X^*.

6.1.2 THEOREM. *If $f:\Omega \to X$ is integrable, then the function $U:X^* \to L^1(\mu)$ given by $U(x^*) = x^* \circ f$ is continuous and linear.*

PROOF. We use the closed graph theorem. Suppose $x_n^* \to x^*$ in X^* and $x_n^* \circ f \to g$ in $L^1(\mu)$. Then $x_n^* \circ f \to x^* \circ f$ pointwise, and some subsequence of $\{x_n^* \circ f\}$ converges a.e. to g (by 2.2.8). Hence $g = x^* \circ f$ a.e., which shows that the graph of U is closed.

6.1.3 COROLLARY. *If $f:\Omega \to X$ is integrable, then for every E in Σ there is a unique element, denoted $(^*)\int_E f\,d\mu$, in X^{**} such that*

$$[(\hat{\ }) \int_E f d\mu](x^*) = \int_E x^* \circ f d\mu, \quad \forall x^* \in X^*.$$

PROOF. $(\hat{\ }) \int_E f\, d\mu$ is the composition $x^* \mapsto x^* \circ f \mapsto \int_E x^* \circ f d\mu$.

6.1.4 If $f:\Omega \to X$ is integrable, $(\hat{\ }) \int_E f d\mu$ is called the (Gelfand or Dunford's second) integral of f over E.

6.1.5 An integrable function $f:\Omega \to X$ is said to be Pettis integrable provided $(\hat{\ }) \int_E f d\mu$ is in $\hat{X}$ for all E in Σ. In this case, the Pettis integral of f over $E \in \Sigma$ is the element, denoted $\int_E f d\mu$, in X satisfying

$$(\int_E f d\mu)\hat{\ } = (\hat{\ }) \int_E f d\mu;$$

i.e.,

$$x^*(\int_E f d\mu) = \int_E x^* \circ f d\mu, \quad \forall x^* \in X^*.$$

6.1.6 THEOREM. If $f:\Omega \to X$ is Pettis integrable and if $\varphi:X \to Y$ is any continuous linear operator, then $\varphi \circ f:\Omega \to Y$ is Pettis integrable, and

$$\int_E \varphi \circ f d = \varphi(\int_E f d\mu), \quad \forall E \in \Sigma.$$

PROOF. If $y^* \in Y^*$, then $y^* \circ \varphi \in X^*$, so $y^* \circ \varphi \circ f$ is in $L^1(\mu)$. Moreover,

$$y^*(\varphi(\int_E f d\mu)) = y^* \circ \varphi(\int_E f d\mu) = \int_E y^* \circ \varphi \circ f d\mu.$$

6.1.7 THEOREM. *If $f: \Omega \to X$ is Pettis integrable, and if*

$$\mu_f(E) = \int_E f d\mu, \quad \forall E \in \Sigma,$$

then $\mu_f: \Sigma \to X$ is countably additive and absolutely continuous with respect to μ.

PROOF. Note that for all x^* in X^*, $x^* \circ \mu_f(E) = \int_E x^* \circ f d\mu$, so $x^* \circ \mu_f$ is c.a. By Pettis' theorem (5.2.4.), μ_f is c.a.

Also, $\mu(E) = 0 \Rightarrow x^* \circ \mu_f(E) = 0$, $\forall x^* \Rightarrow \mu_f(E) = 0$, so μ_f is absolutely continuous with respect to μ by 5.1.6.

6.1.8 LEMMA. *Let $\{g_n\}$ be a sequence in $L^1(\mu)$, and for each n, let $\nu_n(E) = \int_E g_n d\mu$ $(E \in \Sigma)$. If $\{g_n\}$ is Cauchy in measure, the following statements are equivalent.*

(1) $\lim_{n\to\infty} \nu_n(E)$ *exists for all E in Σ.*

(2) $\lim_{n\to\infty} \nu_n(E)$ *exists uniformly over E in Σ.*

(3) $\{\nu_n\}$ *is uniformly absolutely continuous with respect to μ.*

6.1.7.

PROOF. (1) ⇒ (3) by the Vitali-Hahn-Saks theorem, and clearly (2) ⇒ (1). It remains to show (3) ⇒ (2).

Suppose (3) holds and let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$\mu(E) < \delta \Rightarrow \|\nu_n(E)\| < \varepsilon, \quad \forall_n.$$

Now since $\{g_n\}$ is Cauchy in μ-measure, $\exists N$ such that

$$\forall m,n \geq N \ \exists E_{mn} \in \Sigma \ \text{ with } \ \mu(E_{mn}) < \delta \quad \text{and}$$

$$\omega \notin E_{mn} \Rightarrow |g_n(\omega) - g_m(\omega)| < \frac{\varepsilon}{\mu(\Omega)+1}\,.$$

Then for any E in Σ, $m,n \geq N$ implies

$$\begin{aligned}|\nu_n(E) - \nu_m(E)| &\leq |\nu_n(E \cap E_{mn})| + |\nu_m(E \cap E_{mn})| \\ &\quad + |\nu_n(E\setminus E_{mn}) - \nu_m(E\setminus E_{mn})| \\ &< \varepsilon + \varepsilon + \left|\int_{E\setminus E_{mn}} (g_n - g_m)\,d\mu\right| \\ &\leq 2\varepsilon + \frac{\varepsilon}{\mu(\Omega)+1}\,\mu(E\setminus E_{mn}) \leq 3\varepsilon.\end{aligned}$$

6.1.9 LEMMA (Vitali). Let $\{g_n\}$ be a sequence in $L^1(\mu)$, and let g be a measurable function. In order that g be in $L^1(\mu)$

<u>and</u> $\|g_n - g\|_1 \to 0$, <u>it is necessary and sufficient that</u>

(a) $g_n \to g$ <u>in measure</u>

<u>and</u> (b) <u>one</u> (<u>and hence all</u>) <u>of</u> (1), (2), (3) <u>of</u> 6.1.8 <u>hold</u>.

PROOF. ($\Rightarrow$) Trivial.

($\Leftarrow$) Suppose $g_n \to g$ in measure and $\{\nu_n\}$ is uniformly absolutely continuous with respect to μ. For given $\varepsilon > 0$, let $G_{mn} = \{\omega: |g_n(\omega) - g_m(\omega)| \geq \varepsilon\}$. Then $\mu(G_{mn}) \to 0$ as $m,n \to \infty$, and so there exists N such that

$$m,n \geq N \Rightarrow \int_{G_{mn}} |g_k|\,d\mu < \varepsilon, \quad \forall k.$$

Hence for all $m,n \geq N$

$$\begin{aligned}\int |g_m - g_n|\,d\mu &\leq \int_{G_{mn}} |g_m|\,d\mu + \int_{G_{mn}} |g_n|\,d\mu + \int_{\Omega\setminus G_{mn}} |g_m - g_n|\,d\mu \\ &< \varepsilon + \varepsilon + \varepsilon\mu(\Omega\setminus G_{mn}) \\ &\leq 2\varepsilon + \varepsilon\mu(\Omega).\end{aligned}$$

This shows that $\{g_n\}_1^\infty$ is Cauchy in mean and hence converges in mean.

6.1.10 A sequence $\{f_n\}$ of weakly measurable functions on Ω to X is said to converge weakly in measure to $f:\Omega \to X$ if and only if $\{x^* \circ f_n\}$ converges in measure to $x^* \circ f$ for all x^* in X^*.

6.1.11 THEOREM. Let $\{f_n\}_1^\infty$ be a sequence of Pettis integrable functions and suppose

(1) $f_n \to f$ weakly in measure,

and (2) $\text{wk} - \lim\limits_{n\to\infty} \int_E f_n d\mu$ exists for all E in Σ.

Then f is Pettis integrable and

$$\int_E f d\mu = \text{wk} - \lim_{n\to\infty} \int_E f d\mu.$$

[The weak limits in (2) and (3) can simultaneously be replaced by strong limits.]

PROOF. For every x^* in X^*, $x^* \circ f_n \to x^* \circ f$ in measure, so f is weakly measurable. Also, since

$$\lim_n \int_E x^* \circ f_n d\mu = x^*(\text{wk} - \lim \int_E f_n d\mu)$$

exists for all E in Σ, by Vitali's lemma, $x^* \circ f_n$ converges in the L^1-norm to $x^* \circ f$. Thus f is Pettis integrable with

$$\int_E f d\mu = \mathrm{wk} - \lim \int_E f_n d\mu.$$

6.1.12 In the space of Pettis integrable functions we can introduce the following pseudo-norm:

$$\|f\|_P = \sup_{\|x^*\|\le 1} \|x^* \circ f\|_1.$$

Thus, $\|f\|_P$ is the operator norm of the operator U in 6.1.2.

6.1.13 THEOREM. Let $\{f_n\}$ be a sequence of Pettis integrable functions which is $\|\cdot\|_P$-Cauchy. Then $\{f_n\}$ $\|\cdot\|_P$-converges to a Pettis integrable function if and only if it converges weakly in measure to some weakly measurable function.

PROOF. ($\Rightarrow$) If $f_n \to f$ in the $\|\cdot\|_P$ norm, then for all x^*, $x^* \circ f_n \to x^* \circ f$ in $L^1(\mu)$, so $x^* \circ f_n \to x^* \circ f$ in measure.

($\Leftarrow$) Suppose $\{f_n\}$ is $\|\cdot\|_P$-Cauchy and $f_n \to f$ weakly in measure, where f is some weakly measurable function. The hypotheses of 6.1.11 are easily checked, so f is Pettis integrable. Now given $\varepsilon > 0$ there exists N such that

$$m,n \geq N \Rightarrow \sup_{\|x^*\|\leq 1} \|x^* \circ f_m - x^* \circ f_n\|_1 < \varepsilon,$$

and so

$$n \geq N \Rightarrow \sup_{\|x^*\|\leq 1} \|x^* \circ f_m - x^* \circ f\|_1 \leq \varepsilon,$$

since $x^* \circ f \to x^* \circ f$ in $L^1(\mu)$ and the norm in $L^1(\mu)$ is continuous. Thus $f_n \to f$ in the $\|\cdot\|_P$-norm.

6.2 INTEGRALS OF STRONGLY MEASURABLE FUNCTIONS.

6.2.1. A function $f: \Omega \to X$ is <u>strongly measurable</u> if and only if there exists a sequence of simple functions $f_n: \Omega \to X$ such that $f_n(\omega) \to f(\omega)$ in norm for μ-almost all ω in Ω.

A function $f:\Omega \to X$ is said to be <u>essentially separably valued</u> if and only if there exists $\Omega_1 \in \Sigma$ such that $\mu(\Omega \setminus \Omega_1) = 0$ and $f(\Omega_1)$ lies in a separable subspace of X. Clearly a strongly measurable function is essentially separably valued and also weakly measurable. We prove below the converse of this, a powerful result due to Pettis.

A function $f:\Omega \to X$ having a countable range will be called <u>countably valued</u>.

6.2.2. LEMMA. <u>If $f:\Omega \to X$ is weakly measurable and essentially separably valued, then $\|f\|:\Omega \to \mathbb{R}$ is measurable. In particular, if f is strongly measurable, then $\|f\|$ is measurable.</u>

PROOF. We may assume without loss of generality that X is separable. Let D be a countable weak*-dense subset of the unit ball B^* of X^*. (B^* is a compact metric space by).

For any α in $\mathbb{R}$,

$$\{\omega: \|f(\omega)\| \leq \alpha\} = \bigcap_{x^*\in B^*} \{\omega : |x^*(f(\omega))| \leq \alpha\}$$

$$= \bigcap_{x^*\in D} \{\omega : |x^*(f(\omega))| \leq \alpha\},$$

6.2.1

which is in Σ since D is countable and f is weakly measurable. Thus $\|f\|$ is measurable.

6.2.3. THEOREM (Egoroff) Let f and $f_n (n = 1,2,\ldots)$ be strongly measurable functions on Ω to X. Then the following two statements are equivalent.

(1) $f_n \to f$ a.e. in norm.

(2) $\forall \epsilon > 0 \ \exists E \in \Sigma$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $\Omega \setminus E$.

PROOF. By translation we may assume $f = 0$. By 6.2.2, $\|f_n\|$ is measurable for every n. The proof now proceeds exactly as in 2.2.6.

6.2.4. THEOREM (Pettis). For a function $f: \Omega \to X$, the following three statements are equivalent.

(1) f is strongly measurable.

(2) f is weakly measurable and essentially separably valued.

(3) f can be approximated uniformly, except possibly on a set of measure zero, by countably-valued strongly measurable functions.

PROOF. (1) $\Rightarrow$ (2) trivially.

(2) $\Rightarrow$ (3). We assume without loss if generality that X is separable. Let $\epsilon > 0$ be given. If each $f(\omega)$ is covered by an open ϵ-ball in X, then by Lindelof's theorem, a countable collection $\{B_i\}_{i=1}^{\infty}$ of these balls

covers $f(\Omega)$. Let x_i be the center of B_i. Then $f(.) - x_i$ is weakly measurable and by 6.2.2, $\|f(.) - x_i\|$ is measurable. Hence the set

$$E_i = \{\omega : f(\omega) \in B_i\}$$

is in Σ, and $\Omega = \bigcup_{i=1}^{\infty} E_i$. Let $F_1 = E_1$, $F_2 = E_2 \setminus E_1, \ldots$ $F_k = E_k \setminus (\bigcup_{i=1}^{k-1} E_i), \ldots,$ and let $g(\omega) = x_i$ if and only if $\omega \in F_i$. Then $\|f(\omega) - g(\omega)\| < \epsilon$ for all ω since $\Omega = \bigcup_{i=1}^{\infty} F_i$. Clearly g is strongly measurable and countably-valued.

(3) $\Rightarrow$ (1). If (3) holds, then by Egoroff's theorem we can choose a simple function g_n such that $\|g_n(\omega) - f(\omega)\| < \frac{1}{2^n}$ for all ω except in a set $A_n \in \Sigma$ with $\mu(A_n) < \frac{1}{2^n}$. Then $\mu(\bigcap_{m=1}^{\infty} (\bigcup_{n=m}^{\infty} A_n)) = 0$ and if $\omega \notin \bigcap_{m=1}^{\infty} (\bigcup_{n=m}^{\infty} A_n)$ we have that $\exists M \ni m \geq M$ implies $\omega \notin A_m$ so $\|g_m(\omega) - f(\omega)\| < \frac{1}{2^m}$. Hence f is strongly measurable.

6.2.5. COROLLARY. _If X is separable, then $f: \Omega \to X$ is strongly measurable if and only if it is weakly measurable._

6.2.6. COROLLARY. _If (f_n) is a sequence of strongly measurable functions and if $f_n(\omega) \to f(\omega)$ weakly for almost all $\omega \in \Omega$, then f is strongly measurable._

6.2.5.

we can assume without loss of generality that X is separable and hence there exists a countable weak*-dense subset D of B^*. Then

$$E_1 = \{\omega : f(\omega) = x_1\}$$

$$= \{\omega : x^*(f(\omega)) = x^*(x_1), \ \forall x^* \in D\}$$

$$= \bigcap_{x^* \in D} \{\omega : x^*(f(\omega)) = x^*(x_1)\}.$$

which must be in Σ.

6.2.8. THEOREM. If f and g are strongly measurable integrable functions, then $f = g$ a.e. if and only if

$$(\wedge)\int_E f d\mu = (\wedge)\int_E g d\mu \quad , \forall E \in \Sigma$$

PROOF. ($\Rightarrow$) clearly.

($\Leftarrow$). Suppose $(\wedge)\int_E f d\mu = (\wedge)\int_E g d\mu$ for all $E \in \Sigma$. We may assume X is separable and that $D = \{x_n^*\}_1^\infty$ is a countable weak*-dense subset of B^*. Since $\int_E x_n^* \circ f \, d\mu = \int_E x_n^* \circ g d\mu$, $\forall E \in \Sigma$, we must have $x_n^* \circ f(\omega) = x_n^* \circ g(\omega)$ for all ω except in some set $A_n \in \Sigma$ with $\mu(A_n) = 0$. Let $A = \bigcup_1^\infty A_n$. Then $\mu(A) = 0$ and if ω is not in A,

$$x_n^*(f(\omega)) = x_n^*(g(\omega))$$

for all n, so $f(\omega) = g(\omega)$.

6.2.9. LEMMA. *A uniformly bounded strongly measurable function* $f : \Omega \to X$ *is Pettis integrable.*

PROOF. We may assume X is separable and so the unit ball B^* of X^* is metric in the weak* topology. Clearly f is integrable and it is only necessary to show that $x^* \to \int_E x^* \circ f d\mu$ is weak*-continuous on B^*. Suppose $x_n^* \to x^*$ weak* in B^*. Then $x_n^* \circ f \to x^* \circ f$ pointwise and by Lebesgue's dominated convergence theorem $\int_E (x_n^* \circ f) d\mu \to \int_E (x^* \circ f) d\mu$.

6.2.10. THEOREM. *Let* $f : \Omega \to X$ *be strongly measurable and let*

$$f = g + \sum_{i=1}^{\infty} x_i \chi_{E_i}.$$

be any representation as in 6.2.7.

The function f *is integrable if and only if for each* E *in* Σ *and* x^* *in* X^* *the series*

$$\sum_{i=1}^{\infty} x^*(x_i)\mu(E \cap E_i)$$

converges absolutely. In this case

$$(\cdot)\int_E f d\mu = (\cdot)\int_E g d\mu + (wk^*\text{-}\sum_{i=1}^{\infty} \hat{x}_i \mu(E \cap E_i))$$

The function f *is Pettis integrable if and only if for each* E *in* Σ*, the series*

$$\sum_{i=1}^{\infty} x_i \, \mu(E \cap E_i)$$

converges unconditionally. In this case

$$\int_E f d\mu = \int_E g d\mu + \sum_{i=1}^{\infty} x_i \mu(E \cap E_i).$$

PROOF. If f is integrable, then for every E in Σ and x^* in X^*,

$$\left[(\sim)\int_E f d\mu\right](x^*) = \int_E x^* \circ f \, d\mu$$

$$= \int_E x^* \circ g d\mu + \sum_{i=1}^{\infty} x^*(x_i)\mu(E \cap E_i)$$

by the dominated convergence theorem. Since the last series is unconditionally convergent, it is absolutely convergent.

Conversely, suppose for every E in Σ and x^* in X^* the series $\Sigma\, x^*(x_i)\mu(E \cap E_i)$ is absolutely convergent. Note that for every x^* in X^*,

$$x^* \circ f = x^* \circ g + \Sigma\, x^*(x_i)\, \chi_{E_i}.$$

Since $x^* \circ g$ is bounded, $x^* \circ g$ is in $L^1(\mu)$. If E in Σ, then

$$\int_E \Big(\sum_{i=1}^{n} x^*(x_i)\chi_{E_i}\Big)d\mu = \sum_{i=1}^{n} x^*(x_i)\mu(E_i \cap E) \to \sum_{i=1}^{\infty} x^*(x_i)\mu(E \cap E).$$

By 3.2.11. and the Radon-Nikodym Theorem, $\sum_{i=1}^{n} x^*(x_i)\, \chi_{E_i}$ converges weakly to some member of $L^1(\mu)$.

Hence f is integrable. Since the series $\sum_{i=1}^{\infty} \hat{x}_i \mu(E \cap E_i)$ converges at every x^* in X^*, it is bounded by the Uniform Boundedness Principle, and hence it converges weak* using Alaoglu's theorem, and it follows that

$$(\hat{\ })\int_E f d\mu = (\hat{\ })\int_E g d\mu + (\text{wk}^*\text{-}\sum_{i=1}^{\infty} x_i \mu(E \cap E_i)).$$

If f is Pettis integrable, then $\sum_{i=1}^{\infty} x_i \chi_{E_i}$ is Pettis integrable by 6.2.9. Moreover,

$$\begin{aligned} x^*(\int_E (\sum_{i=1}^{\infty} x_i \chi_{E_i}) d\mu) &= \int_E (\sum_{i=1}^{\infty} x^*(x_i) \chi_{E_i}) d\mu \\ &= \sum_{i=1}^{\infty} x^*(x_i) \mu(E \cap E_i) \\ &= x^*(\sum_{i=1}^{\infty} x_i \mu(E \cap E_i)), \end{aligned}$$

the last equality holding by the Pettis-Orlicz theorem.

Conversely, if $\sum_{i=1}^{\infty} x_i \mu(E \cap E_i)$ is unconditionally convergent for all E, then f is integrable by the above and

$$\begin{aligned} (\hat{\ })\int_E f d\mu &= (\hat{\ })\int_E g d\mu + (\text{wk}^*\text{-}\sum_{i=1}^{\infty} \hat{x}_i \mu(E \cap E_i)) \\ &= (\int_E g d\mu) + (\sum_{i=1}^{\infty} x_i \mu(E \cap E_i))^{\wedge} \end{aligned}$$

is in $\hat{X}$, so f is Pettis integrable.

6.2.10.

6.2.11. EXAMPLE. Let $f : \mathbb{N} \to c_o$ be defined by $f(n) = 2^n e_n$, where $e_n(m) = \delta_{mn}$. Let $\mu \in ca(2^{\mathbb{N}})$ be such that $\mu(\{n\}) = \frac{1}{2^n}$. Then f is strongly measurable and countably-valued, with

$$f = \sum_{i=1}^{\infty} 2^i e_i \chi_{\{i\}}.$$

For any set $E \subset \mathbb{N}$,

$$\sum_{i=1}^{\infty} 2^i e_i \mu(\{i\} \cap E) = \sum_{i \in E} e_i.$$

If E is infinite, this series does not converge in c_o and so f is not Pettis integrable. However, for every $x^* \in c_o^* = \ell^1$, say $x^* = (\alpha_i)_{i=1}^{\infty}$,

$$\sum_{i \in E} x^*(e_i) = \sum_{i \in E} \alpha_i$$

converges absolutely. Hence f is Gelfand integrable with

$$(\sim)\int_E f d\mu = wk^* - \sum_{i \in E} e_i = \chi_E \in (c_o)^{**} = (\ell_1)^* = \ell_\infty.$$

6.2.12. THEOREM. <u>For a function $f:\Omega \to X$, the following three statements are equivalent.</u>

(1) <u>f is strongly measurable and Pettis integrable.</u>

(2) <u>There exists a sequence of simple functions $\{f_n\}$ such that $f_n \to f$ a.e. and $\|f - f_n\|_P \to 0$.</u>

(3) <u>There exists a sequence of simple functions $\{f_n\}$ such that $f_n \to f$ a.e. and</u>

$$\lim_n \int_E f_n \, d\mu$$

<u>exists for every E in Σ.</u>

PROOF. (1) $\Rightarrow$ (2). Using 6.2.7 write

$$f = g + h = g + \left(\sum_{i=1}^{\infty} x_i \chi_{E_i}\right), \quad \text{a.e.}$$

Let $h_n = \sum_{i=1}^{n} x_i \chi_{E_i}$. Then $h_n \to h$ everywhere and

$$\|h - h_n\|_{\mathcal{P}} = \sup_{\|x^*\| \leq 1} \|x^*(h - h_n)\|_1$$

$$\leq 2 \sup_{\|x^*\| \leq 1} \sup_{E \in \Sigma} \left| \int_E x^*(h - h_n)\,d\mu \right|$$

$$= 2 \sup_{E \in \Sigma} \sup_{\|x^*\| \leq 1} \left| x^*\left(\int_E (h - h_n)\,d\mu\right)\right|$$

$$= 2 \sup_{E \in \Sigma} \left\| \int_E (h - h_n)\,d\mu \right\|$$

$$= 2 \sup_{E \in \Sigma} \left\| \int_{E \cap (\bigcup_{i=n+1}^{\infty} E_i)} h\,d\mu \right\|$$

$$\to 0$$

since $\lim_{\mu(F) \to 0} \left\| \int_F h\,d\mu \right\| = 0$ (by 6.1.7) and $\mu\left(\bigcup_{i=n+1}^{\infty} E_i\right) = 0$

Next we can choose a sequence (g_n) of simple functions such that $g_n \to g$ a.e., and since g is bounded we may assume $\|g_n\| \leq M$ for all n, some M. Then by the dominated convergence theorem

$$\|g - g_n\|_{p} = \sup_{\|x^*\| \leq 1} \int |x^*(g - g_n)|\,d\mu \leq \int \|g_n - g\|\,d\mu \to 0.$$

Letting $f_n = h_n + g_n$ we have $f_n \to f$ a.e. and $\|f - f_n\|_{\mathcal{P}} \to 0$.

(2) ⇒ (3). If $\|f_n - f\|_{\mathcal{P}} \to 0$, then for any E in Σ

$$\left\| \int_E f_n d\mu - \int_E f_m d\mu \right\| = \left\| \int_E (f_n - f_m) d\mu \right\|$$

$$= \sup_{\|x^*\| \leq 1} \left| \int_E x^*(f_n - f_m) d\mu \right|$$

$$\leq \sup_{\|x^*\| \leq 1} \int |x^*(f_n - f_m)| d\mu$$

$$= \|f_n - f_m\|_{\mathcal{P}}$$

$$\to 0$$

as $m, n \to \infty$.

(3) ⇒ (1). If $f_n \to f$ a.e., then f_n converges to f weakly in measure. Thus if $\lim \int_E f_n d\mu$ also exists for all E in Σ, then by 6.1.11, f is Pettis integrable.

6.2.13. THEOREM. Let $f : \Omega \to X$ be strongly measurable and integrable. Let $\nu(E) = (\hat{\hat{w}}) \int_E f d\mu \in X^{**}$, $(E \in \Sigma)$. Then the following four statements are equivalent.

(1) f is Pettis integrable.

(2) $K = \{\hat{x^*} \circ \nu \ : \ x^* \in X^*, \|x^*\| \leq 1\}$ is conditionally weakly compact in $ca(\Sigma)$.

(3) ν is countably additive.

(4) ν is absolutely continuous with respect to μ.

PROOF. (1) $\Rightarrow$ (3) by 6.1.7.

(3) $\Rightarrow$ (2). If ν is c.a., then by

$$\{x^{***} \circ \nu \; : \; x^{***} \in X^{***} \, , \; \|x^{***}\| \leq 1\}$$

is conditionally weakly compact and it contains K.

(2) $\Rightarrow$ (4). Suppose (2) holds. Note that μ is a control measure for K. Hence by 3.3.1, μ is a uniform control measure. If $\mu(E_n) \to 0$, then

$$\|\nu(E_n)\| = \sup_{\|x^*\| \leq 1} |\nu(E_n)(x^*)| \to 0.$$

(4) $\Rightarrow$ (1). Since f is strongly measurable, we can assume X is separable and therefore B^* is matrizable in the weak* topology. Suppose $x_n^* \to x^*$ weak* in B^*. If (4) holds, then $\{x_n^* \circ \nu\}$ is uniformly absolutely continuous with respect to μ; i.e., $\{\int_{(\cdot)} (x^* \circ f)\, d\mu\}_1^\infty$ is uniformly absolutely continuous with respect to μ. Also, $x_n^* \circ f \to x^* \circ f$ pointwise. Thus by 6.1.9, $x_n^* \circ f \to x^* \circ f$ in $L^1(\mu)$, and so for any E in Σ,

$$((\wedge)\int_E f d\mu)(x_n^*) = \int_E x_n^* \circ f d\mu \to \int_E x^* \circ f \, d\mu = ((\wedge)\int_E f d\mu)(x^*).$$

Therefore f is Pettis integrable.

6.2.14. THEOREM. <u>If $f : \Omega \to X$ is strongly measurable, the following three statements are equivalent.</u>

(1) Whenever f is represented by

(*) $$f = g + \sum_{i=1}^{\infty} x_i \chi_{E_i} \quad \text{a.e.}$$

as in 6.2.7, the series $\sum_{i=1}^{\infty} \|x_i\|\mu(E_i)$ is convergent.

(2) $\int \|f\| d\mu < \infty$

(3) There exists a sequence (f_n) of simple functions such that $f_n \to f$ a.e. and

$$\int \|f - f_n\| d\mu \to 0.$$

PROOF. (1)$\Rightarrow$(2). If (1) holds, then

$$\begin{aligned}\int \|f\| d\mu &\leq \int \|g\| d\mu + \int \|\sum_i x_i \chi_{E_i}\| d\mu \\ &= \int \|g\| d\mu + \int \sum_i \|x_i\| \chi_{E_i} d\mu \\ &= \int \|g\| d\mu + \sum_i \|x_i\| \mu(E_i) \\ &< \infty\end{aligned}$$

(2)$\Rightarrow$(1). In (*), let $h = \sum_{i=1}^{\infty} x_i \chi_{E_i}$. Then $\|h\| = \|f - g\| \leq \|f\| + \|g\|$, so

$$\infty > \int \|h\| d\mu = \sum_{i=1}^{\infty} \|x_i\| \mu(E_i).$$

(2)$\Rightarrow$(3). Choose simple functions g_n such that $g_n \to f$ a.e., and let

$$f_n(\omega) = \begin{cases} g_n(\omega) & \text{if } \|g_n(\omega)\| < 2\,\|f(\omega)\| \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to f$ a.e., and $\|f_n(\omega) - f(\omega)\| \leq 3\|f(\omega)\|$, $\forall\omega$, and so by the dominated convergence theorem

$$\int \|f - f_n\| d\mu \to 0.$$

(3) $\Rightarrow$ (2). For any n, $\int \|f\| d\mu \leq \int \|f - f_n\| d\mu + \int \|f_n\| d\mu < \infty$.

6.2.15. DEFINITION. A function $f:\Omega \to X$ is said to be Bochner integrable if and only if it is strongly measurable and (1), (2), (3) of 6.2.14 are satisfied.

By 6.2.10, every Bochner integrable function is Pettis integrable.

6.2.16. EXAMPLE. Let $f:\mathbb{N} \to c_o$ be given by

$$f(n) = \frac{2^n}{n} \epsilon_n$$

where $\epsilon_n(m) = \delta_{mn}$, and let $\mu \in ca(2^{\mathbb{N}})$ be such that $\mu(\{n\}) = \frac{1}{2^n}$. Then $f = \sum_{i=1}^{\infty} \frac{2^i}{i} \epsilon_i \chi_{\{i\}}$ and

$$\sum_{i=1}^{\infty} \frac{2^i}{i} \epsilon_i \mu(\{i\} \cap E) = \sum_{i \in E} \frac{\epsilon_i}{i} .$$

This converges unconditionally in c_o, and so f is Pettis integrable. However, f is not Bochner integrable since

$$\sum_{i=1}^{\infty} \left\| \frac{\epsilon_i}{i} \right\|_{\infty} = \sum_{i=1}^{\infty} \frac{1}{i} = \infty .$$

6.2.17. THEOREM (Dominated Convergence Theorem).

Let $\{f_n\}$ be a sequence of Bochner integrable functions, and suppose $f_n \to f$ a.e. If there exists some g in $L^1(\mu)$ such that $\|f_n\| \leqq g$ a.e., then f is Bochner integrable and $\int\|f - f_n\|d\mu \to 0$. (In particular, $\int_E f_n d\mu \to \int_E f d\mu$ for all E in Σ).

PROOF. If $f_n \to f$ a.e., then $\|f_n - f\| \to 0$ a.e. and $\|f_n - f\| \leqq 2g$ a.e. Hence $\int\|f_n - f\|d\mu \to 0$ by the scalar dominated convergence theorem.

6.2.18. If ν is in $ca(\Sigma,X)$, the variation of ν over a set E in Σ is defined to be

$$|\nu|(E) = \sup \sum_{i=1}^{n} \|\mu(E_i)\| \leqq +\infty ,$$

where the supremum is taken over all finite disjoint collections $\{E_i\} \subset \Sigma$ with $\cup E_i \subset E$. The measure ν is said to be of bounded variation (b.v.) if and only if $|\nu|(\Omega) < \infty$.

6.2.19. EXERCISE. Show that if ν is of bounded variation, then $|\nu|$ is in $ca^+(\Sigma)$.

6.2.20. THEOREM. Let $f:\Omega \to X$ be strongly measurable and Pettis integrable, and let $\nu(E) = \int_E f d\mu$ $(E \in \Sigma)$. Then f is Bochner integrable if and only if ν is of bounded variation. In that case, $|\nu|(E) = \int_E \|f\| d\mu \quad (E \in \Sigma)$.

PROOF. Suppose f is Bochner integrable, and let $\{E_i\}$ be a finite disjoint collection in Σ.

Then

$$\Sigma\|\nu(E_i)\| = \Sigma\|\int_{E_i} f d\mu\| \leq \Sigma \int_{E_i} \|f\| d\mu \leq \int \|f\| d\mu ,$$

so ν is of b.v. with $|\nu|(\Omega) \leq \int \|f\| d\mu$.

Now fix E in Σ and $\epsilon > 0$. Choose a simple function g such that $\int\|f - g\|d\mu < \epsilon$. Write $g = \sum_{i=1}^{m} x_i \chi_{E_i}$, where the E_i's are disjoint. We may assume that the partition $\{E_1, \ldots, E_n\}$ is so fine that

$$|\nu|(E) - \sum_{i=1}^{n} \|\nu(E_i \cap E)\| < \epsilon .$$

Then

$$\begin{aligned} \Big|\, |\nu|(E) - \int_E \|f\| d\mu \,\Big| &\leq \Big|\, |\nu|(E) - \sum_i \|\nu(E_i \cap E)\| \,\Big| \\ &\quad + \Big|\, \sum_i \|\nu(E_i \cap E)\| - \sum_i \|\int_{E \cap E_i} g d\mu\| \,\Big| \\ &\quad + \Big|\, \sum_i \|\int_{E \cap E_i} g d\mu\| - \int_E \|g\| d\mu \,\Big| \\ &\quad + \Big|\int_E \|g\| d\mu - \int_E \|f\| d\mu\Big| \\ &< \epsilon + \Big|\, \sum_i \|\int_{E \cap E_i} f d\mu\| - \sum_i \|\int_{E \cap E_i} g d\mu\| \,\Big| + 0 + \epsilon \end{aligned}$$

$$\leq 2\epsilon + \sum_i \left| \left\| \int_{E \cap E_i} f d\mu \right\| - \left\| \int_{E \cap E_i} g d\mu \right\| \right|$$

$$\leq 2\epsilon + \sum_i \left\| \int_{E \cap E_i} (f - g) d\mu \right\|$$

$$\leq 2\epsilon + \sum_i \int_{E \cap E_i} \|f - g\| d\mu$$

$$= 2\epsilon + \int_E \|f - g\| d\mu$$

$$< 3\epsilon$$

Hence $|\nu|(E) = \int_E \|f\| d\mu$ since $\epsilon > 0$ was arbitrary.

Conversely, suppose ν is of b.v. Write $f = g + \sum_{i=1}^{\infty} x_i \chi_{E_i}$ a.e. as in 6.2.7. Then $\int_{(.)} g d\mu$ is of b.v., so $\int_{(.)} (\sum_{i=1}^{\infty} x_i \chi_{E_i}) d\mu$ must be of b.v. Then

$$\sum_i \|x_i\| \mu(E_i) = \sup_n \sum_{i=1}^{n} \left\| \int_{E_i} (\sum_{i=1}^{\infty} x_i \chi_{E_i}) \, d\mu \right\| < \infty ,$$

and so f is Bochner integrable by definition.

6.2.21. We have observed that Egoroff's theorem from §2.1 carries over to strongly measurable X-valued functions (6.2.3). We now observe that the rest of §2.1 carries over

6.2.21.

similarly. Let $\mathfrak{M}(\Sigma,\mu,X)$ denote the space of all strongly measurable X-valued functions.

A sequence $\{f_n\}$ in $\mathfrak{M}(\Sigma,\mu,X)$ is said to be Cauchy in (μ-) measure (respectively, to converge in (μ-) measure to $f \in \mathfrak{M}(\Sigma, \mu, X)$) if and only if for every $\epsilon > 0$

$$\lim_{m,n\to\infty} \mu\ \{\omega: \ \|f_n(\omega) - f_m(\omega)\| \geqq \epsilon\} = 0$$

(respectively,

$$\lim_{n\to\infty} \mu\ \{\omega: \ \|f_n(\omega) - f(\omega)\| \geqq \epsilon\} = 0.)$$

As in 2.1.1 and 2.1.2, this mode of convergence is equivalent to convergence with respect to a pseudo-metric ρ on $\mathfrak{M}(\Sigma,\mu,X)$ given by

$$\rho(f,g) = \inf_{\epsilon>0} [\epsilon + \mu\{\omega \in \Omega : \|f(\omega)-g(\omega)\| \geqq \epsilon\}] \ .$$

6.2.22. THEOREM. If f, f_n are in $\mathfrak{M}(\Sigma,\mu,X)$ $(n = 1,2,\ldots)$ and if $f_n \to f$ a.e. $[\mu]$, then $f_n \to f$ in μ-measure.

PROOF. Exactly as in 2.1.7.

6.2.23. THEOREM. If $\{f_n\}$ is a sequence in $\mathfrak{M}(\Sigma,\mu,X)$ which is Cauchy in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges a.e.$[\mu]$ to some member of $\mathfrak{M}(\Sigma,\mu,X)$.

PROOF. Exactly as in 2.1.8.

6.2.24. COROLLARY. $m(\Sigma,\mu,X)$ is a complete pseudo-metric space in the topology of convergence in measure.

6.2.25. DEFINITION. If μ is in $ca^+(\Sigma)$ and X is a Banach space, then $L^1(\mu,X)$ denotes the space of all Bochner integrable functions $f:\Omega \to X$ with the norm

$$\|f\|_1 = \int\|f\|d\mu$$

6.2.26. THEOREM. $L^1(\mu,X)$ is a Banach space.

PROOF. Let $\{f_n\}$ be a sequence in $L^1(\mu,X)$ such that $\int\|f_n - f_m\|d\mu \to 0$ as $m,n \to \infty$. Then $\{f_n\}$ is Cauchy in measure, so there exists a strongly measurable function f such that $f_n \to f$ in measure. It follows that $\|f_n\| \to \|f\|$ in measure, and also that $\lim_n \int_E \|f_n\|d\mu$ exists for all E in Σ. Therefore, by 6.1.9, $\|f\|$ is in $L^1(\mu)$, so f is in $L^1(\mu,X)$. By the Vitali-Hahn-Saks theorem,

$$\lim_{\mu(E)\to 0} \int_E \|f_n\|d\mu = 0 \text{ uniformly over } n.$$

Now given $\epsilon > 0$, let $G_n = \{\omega : \|f_n(\omega) - f(\omega)\| \geq \epsilon\}$. Then $\mu(G_n) \to 0$ as $n \to \infty$, and so there exists N such that

$$n \geq N \Rightarrow \begin{cases} \int_{G_n}\|f_k\|d\mu < \epsilon & \text{for all } k \\ \text{and } \int_{G_n}\|f\|d\mu < \epsilon . \end{cases}$$

Hence $n \geqq N$ implies

$$\int \|f_n - f\| d\mu \leqq \int_{G_n} \|f_n\| d\mu + \int_{G_n} \|f\| d\mu + \int_{\Omega \setminus G_n} \|f_n - f\| d\mu$$

$$< \epsilon + \epsilon + \epsilon\mu(\Omega \setminus G_n)$$

$$\leqq \epsilon\,(2 + \mu(\Omega)).$$

Therefore, $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

6.2.27. COROLLARY. A strongly measurable function $f:\Omega \to X$ is in $L^1(\mu, X)$ if and only if there exists a sequence (f_n) of simple functions such that

(i) $f_n \to f$ in measure (respectively, a.e. $[\mu]$),

and (ii) $\|f_n - f_m\|_1 \to 0$ as $m, n \to \infty$.

6.2.28. EXERCISE. Let X be a Banach space such that X^* is separable. Show that a function $g:\Omega \to X^*$ is strongly measurable if and only if $\hat{x} \circ g$ is measurable for every $\hat{x}$ in $\hat{X}$. (Hint: Use and .)

R.6.1. REMARKS AND REFERENCES.

(1) Our notion of "integrable weakly measurable function" and the corresponding integral (with values in X^{**}) was introduced by Dunford and the integral is referred to as Dunford's second integral. (It is also known as the Gelfand integral.) What we call "Pettis integrable functions" were introduced and studied by Pettis [1938]; all the results of this section (except 6.1.8 and 6.1.9) are from Pettis' paper.

(2) We refer to a survey paper by Hildebrandt [1953] for references to early work on these and other integrals.

<u>Additional</u> <u>References</u>: Brooks [1969(b)].

R.6.2. REMARKS AND REFERENCES.

(1) The important theorem 6.2.4 is due to Pettis [1938, Thm. 1.1].

(2) Theorem 6.2.12 is due to Pettis [1938, Thm. 5.1]. Condition (3) of 6.2.12 was used earlier by Dunford to define an integral, and hence the integral of a strongly measurable Pettis integrable function is often called Dunford's first integral.

(3) The Bochner integral was introduced and studied by Bochner [1933].

(4) Again, see the survey paper by Hildebrandt [1953] for references for these and other integrals.

<u>Additional</u> <u>References</u>: Bartle [1956], Diestel [1973(a)], Brooks [1969(b)].