5. VECTOR-VALUED MEASURES

5.1 S-ADDITIVE MEASURES.

5.1.1 Let S be any set, let X be a Banach space, and let $f:S \to X$ be any bounded function.

Let $f^*:X^* \to m(S)$ be the natural "adjoint" defined by

$$[f^*(x^*)](s) = x^*(f(s)), \quad (x^* \in X^*, s \in S);$$

i.e., $f^*(x^*) = x^* \circ f$. Then f^* is continuous and linear with $\|f^*\| = \|f\|_\infty$.

As a bounded linear operator, f^* has an adjoint $f^{**}:m(S) \to X^{**}$ given by

$$[f^{**}(\gamma)](x^*) = \gamma(f^*(x^*))$$

($\gamma \in m(S)^*$, $x^* \in X^*$), i.e., $f^{**}(\gamma) = \gamma \circ f^*$. As for any adjoint, f^{**} is continuous, linear, and $\|f^{**}\| = \|f^*\| = \|f\|_\infty$. Moreover, f^{**} is weak*-to-weak* continuous.

There exist natural maps $(\hat{\ }):S \to m(S)^*$ and $(\hat{\ }):X \to X^{**}$, the second being an isometric isomorphism onto $\hat{X}$ which is weak* dense in X^{**} (see). As for the first, note that for every s in S, $\hat{s}$ is a positive linear functional of norm one on $m(S)$. Moreover, by the separation theorem it is easily seen that $\overline{co}^{wk^*}(\hat{S})$ is precisely the set of all positive linear functionals of norm one on $m(S)$. It follows from that $m(S)^*$ is the linear span of $\overline{co}^{wk^*}(\hat{S})$. Finally, the outside of the following diagram commutes.

$$
\begin{tikzcd}
m(S)^* \arrow[r, "f^{**}"] & X^{**} \\
m(S) & X^* \arrow[l, "f^*"'] \\
S \arrow[r, "f"'] \arrow[uu, bend left, "(\hat{\ })"] & X \arrow[uu, bend right, "(\hat{\ })"']
\end{tikzcd}
$$

5.1.2 THEOREM. Let S, X, and $f:S \to X$ be as above. Then the following four statements are equivalent.

(i) $f(S)$ is conditionally weakly compact in X.

(ii) $f^{**}:m(S)^* \to X^{**}$ takes its range in $\hat{X}$.

(iii) $f^*:X^* \to m(S)$ is weak*-to-weak continuous.

(iv) $\{f^*(x^*): \|x^*\| = 1\}$ is (conditionally) weakly compact in $m(S)$.

PROOF. (i) ⇒ (ii). Assume (i) and let $K = \overline{co}(f(S))$. Then K is weakly compact in X by . Hence $\hat{K}$ is weakly compact in X^{**} and so $\hat{K}$ is weak* compact in X^{**}. For s in S, $f^{**}(s) = f(s)^{\wedge} \in \hat{K}$. Since f^{**} is weak*-weak* continuous,

$$f^{**}(\overline{co}^{wk^*}(\hat{S})) \subset \hat{K} \subset \hat{X}.$$

Since $m(S)^*$ is the linear span of $\overline{co}^{wk^*}(S)$, $f^{**}(m(\hat{S})^*)$ is contained in $\hat{X}$.

(ii) ⇒ (iii). Assume (ii), and suppose $x_\alpha^* \to x^*$ weak* in X^*. Then given γ in $m(S)^*$,

$$\gamma[f^*(x_\alpha^*)] = [f^{**}(\gamma)](x_\alpha^*) \to [f^{**}(\gamma)](x^*) = \gamma[f^*(x^*)],$$

so $f^*(x_\alpha^*) \to f^*(x^*)$ weakly in $m(S)$.

(iii) ⇒ (iv) by Alaoglu's theorem.

(iv) ⇒ (i). Let $\{s_n\}_1^\infty \subset S$ and $\{x_m^*\}_1^\infty \subset X^*$ be sequences with $\|x_m^*\| \leq 1$, $\forall m$, and suppose both iterated limits

$$\lim_m \lim_n x_m^*(f(s_n))$$

and

$$\lim_n \lim_m x_m^*(f(s_n))$$

exist. Since $x_m^*(f(s_n)) = s_n(f^*(x_m^*))$, if (iv) holds the two limits must agree (by the Eberlein-Smulian theorem). But if the two limits must always agree, then (i) must hold (again by).

5.1.3 REMARK. It is clear that the theorem remains valid if $m(S)$ is replaced by any closed subspace Y for which $x^* \circ f$ is in Y for all x^* in X^*. Moreover, Y can then be given any equivalent norm.

5.1.4 If $\mu : A \to X$ is a finitely additive measure, then a <u>control measure</u> for μ is a member λ of $ba(A)$ such that μ is absolutely continuous with respect to $|\lambda|$.

5.1.5 COROLLARY. <u>Let</u> $\mu : A \to X$ <u>be finitely additive. Then the following five statements are equivalent.</u>

(i) $\mu(A)$ <u>is conditionally weakly compact in</u> X.

(ii) $K = \{x^* \circ \mu : \|x^*\| \leq 1\}$ <u>is (conditionally) weakly compact in</u> $ba(A)$.

(iii) <u>The function</u> $x^* \mapsto x^* \circ \mu$ <u>is weak*-to-weak continuous on</u> X^* <u>to</u> $ba(A)$.

(iv) μ is s-additive on A.

(v) There exists a control measure for μ in $ba(A)$.

Moreover, if (i) - (v) hold, then the set

$$A = \{x^* \in X^* : \|x^*\| \le 1, \ x^* \circ \mu \text{ is a control measure for } \mu\}$$

is dense in $\{x^* \in X^* : \|x^*\| \le 1\}$.

PROOF. We first observe that each of (i) - (v) implies that μ is bounded. (i) and (ii) do by the uniform boundedness principle, and (iii) $\Rightarrow$ (ii). Also (iv) implies μ is bounded by 4.2.7, and (v) $\Rightarrow$ (iv). Hence simply assume μ is bounded.

Since μ is bounded, (i), (ii), and (iii) are equivalent by 5.1.2 (5.1.3).

By 4.4.4, (ii) holds if and only if K is uniformly s-additive, i.e., if and only if given a disjoint sequence $\{E_n\}$ in A and $\varepsilon > 0$ there exists N such that

$$n \geq N \Rightarrow |x^* \circ \mu(E_n)| \leq \varepsilon , \quad \forall \|x^*\| \leq 1$$

$$\Rightarrow \|\mu(E_n)\| \leq \varepsilon.$$

Hence (ii) $\Rightarrow$ (iv), and similarly (iv) $\Rightarrow$ (ii).

Again by 4.4.4, (ii) holds if and only if there exists a uniform control measure λ for K; i.e., there exists λ in $ba(A)$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$|\lambda|(E) < \delta \Rightarrow |x^* \circ \mu(E)| \leq \varepsilon, \quad \forall \|x^*\| \leq 1$$

$$\Rightarrow \|\mu(E)\| \leq \varepsilon.$$

Hence (ii) $\Rightarrow$ (v), and similarly (v) $\Rightarrow$ (ii).

The last statement of the theorem follows from 4.4.7.

5.1.6 COROLLARY. If $\mu:\Sigma \to X$ is countably additive, then the following statements hold.

(a) $\mu(\Sigma)$ is conditionally weakly compact in X.

(b) $K = \{x^* \circ \mu : \|x^*\| \leq 1\}$ is weakly compact in $ca(\Sigma)$.

(c) The function $x^* \to x^* \circ \mu$ is weak*-to-weak continuous on X^* to $ca(\Sigma)$.

(d) $\{x^* \in X^* : \|x^*\| \leq 1$ and $x^* \circ \mu$ is a control measure for $\mu\}$ *is dense in the unit ball of* X^*.

(e) *If* $\lambda \in ca(\Sigma)$ *is such that*

$$|\lambda|(E) = 0 \Rightarrow \mu(E) = 0,$$

then μ *is absolutely continuous with respect to* λ.

PROOF. If $\mu:\Sigma \to X$ is countably additive, then μ is s-additive. Hence (a) - (d) hold by 5.1.5. If $\lambda \in ca(\Sigma)$ is such that $|\lambda|(E) = 0 \Rightarrow \mu(E) = 0$, then λ is a control measure for K, and hence a uniform control measure.

5.1.7 COROLLARY (Nikodým). *Suppose* $\mu_n:\Sigma \to X$ *is countably additive for all* n, *and suppose* $\mu(E) = \lim_{n\to\infty} \mu_n(E)$ *exists for every* E *in* Σ. *Then* $\{\mu_n\}$ *is uniformly countably additive and* μ *is countably additive.*

PROOF. For each n, choose λ_n in $ca(\Sigma)$ such that μ_n is λ_n-continuous. Let

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\lambda_n|}{1 + \|\lambda_n\|}.$$

Then each μ_n is λ-continuous and so the Vitali-Hahn-Saks theorem (3.2.8) applies.

5.1.8 In Chapter 4 we were able to carry over many of the results from Chapter 3 about c.a. measures on σ-algebras to s-additive measures on algebras. Example 4.4.8 shows however that such results as the fundamental Vitali-Hahn-Saks theorem do not carry over. Note that we weakened two properties: (i) the domains were only algebras, and (ii) the measures were only finitely additive. For the remainder of this section we consider the intermediate stage of considering finitely additive measures on σ-algebras. We shall see that the Vitali-Hahn-Saks theorem and related results remain valid in this setting. The key result is the following . (as always, Σ is σ-algebra of subsets of Ω, and X is a Banach space.)

5.1.9 THEOREM. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of s-additive finitely additive measures on Σ to X, and let $\{E_m\}_{m=1}^{\infty}$ be a disjoint sequence in Σ. Then there exists a subsequence $\{E_{m_k}\}_{k=1}^{\infty}$ of $\{E_m\}_{m=1}^{\infty}$ such that every λ_n is countably additive on the σ-algebra $\Sigma(\{E_{m_k}\}_{k=1}^{\infty})$ generated by $\{E_{m_k}\}_{k=1}^{\infty}$.

PROOF. By passing to control measures, we may assume without loss of generality that the λ_n's are real-valued and non-negative.

Let (A_0, φ) denote the Stone Completion of Σ (see 4.3). Let $\mathfrak{A}$ be an uncountable collection of infinite subsets of $\mathbb{N}$ such that any two members of $\mathfrak{A}$ have finite intersection. (The existence of such a collection can be proved by enumerating the rationals and taking, for each irrational α, a sequence of distinct rationals converging to α.) For every A in $\mathfrak{A}$, let

$$C_A = \bigcap_{k=1}^{\infty} [\varphi(\bigcup_{\substack{j \geq k \\ j \in A}} E_j)].$$

Then $A, B \in \mathfrak{A}$, $A \neq B$, implies $C_A \cap C_B \neq \phi$. Hence there exists some A in $\mathfrak{A}$ such that

$$\overline{\lambda_n \circ \varphi^{-1}}(C_A) = 0, \qquad \forall_n,$$

where $\overline{\lambda_n \circ \varphi^{-1}}$ denotes the unique extension of $\lambda_n \circ \varphi^{-1}$ to a countably additive measure on $\Sigma(A_0)$. Let $A = \{m_1 < m_2 < \ldots\}$ and consider the subsequence $\{E_{m_k}\}_{k=1}^{\infty}$ of $\{E_m\}_{m=1}^{\infty}$. We have

$$\lim_{k \to \infty} \lambda_n (\bigcup_{j=k}^{\infty} E_{m_j}) = \lim_{k \to \infty} \overline{\lambda_n \circ \varphi^{-1}} [\varphi(\bigcup_{j=k}^{\infty} E_{m_j})]$$
$$= \overline{\lambda_n \circ \varphi^{-1}}(C_A) = 0,$$

for every n, and it follows easily that λ_n is countably additive on $\Sigma(\{E_{n_j}\}_{j=1}^{\infty})$.

5.1.10 THEOREM. Let K be a set of finitely additive measures $\mu:\Sigma \to X$, and suppose K is elementwise bounded on Σ; i.e., for every E in Σ

$$\sup_{\lambda\in K} \|\lambda(E)\| < \infty.$$

Then K is uniformly bounded on Σ; i.e.,

$$\sup_{E\in\Sigma}\sup_{\lambda\in K} \|\lambda(E)\| < \infty.$$

PROOF. If K is not uniformly bounded, then neither is the set $\{x^* \circ \lambda : \lambda \in K, x^* \in X^*, \|x^*\| \le 1\}$. Thus we may assume without loss of generality that the measures are real-valued.

Define η on Σ to $[0,\infty]$ by

$$\eta(E) = \sup_{\substack{F\subset E\\ F\in\Sigma}}\sup_{\lambda\in K} |\lambda(F)|.$$

Note that $\lambda(E_1 \cup E_2) \le \eta(E_1) + \eta(E_2)$ for all E_1, E_2 in Σ. We are to show that $\eta(\Omega) < \infty$.

We claim that if $\eta(E) = \infty$ and $M \in \mathbb{R}$, then there exists F in Σ, $F \subset E$, and $\lambda \in K$ such that $|\lambda(F)| > M$ and $\eta(E\setminus F) = \infty$. For, if $\eta(E) = \infty$, then there exists $G \subset E$ and λ in K such that

$$|\lambda(G)| > M + \sup_{\mu \in K} |\mu(E)|.$$

Now $\infty = \eta(G \cup (E \backslash G)) \leq \eta(G) + \eta(E \backslash G)$, so that either $\eta(G) = \infty$ or $\eta(E \backslash G) = \infty$. If $\eta(E \backslash G) = \infty$, take $F = G$. If instead $\eta(G) = \infty$, take $F = E \backslash G$, and note that

$$|\lambda(F)| = |\lambda(E) - \lambda(G)| \geq |\lambda(G)| - |\lambda(E)| \geq M.$$

Hence the claim holds.

Now by the above, if $\eta(\Omega) = \infty$, we can inductively define a disjoint sequence $\{E_n\}$ in Σ and a sequence $\{\lambda_n\}$ in K such that $|\lambda_n(E_n)| > n$ for every n. By 5.1.9, we may assume that every λ_n is countably additive on $\Sigma_0 = \Sigma(\{E_n\}_{n=1}^{\infty})$. But then by 3.2.10, $\{\lambda_n\}$ is uniformly bounded on Σ_0, a contradiction. Therefore we must have that $\eta(\Omega) < \infty$.

5.1.11 THEOREM. Let $\{\overset{\mu_n}{\lambda_n}\}$ be a sequence of s-additive finitely additive measures on Σ to X, and suppose that for every E in Σ,

$$\mu(E) = \lim_{n \to \infty} \mu_n(E)$$

exists. Then $\{\mu_n\}$ is uniformly s-additive and μ is s-additive.

PROOF. Let $K = \{\mu_n : n = 1,2,\ldots\}$. If K is not uniformly s-additive, then there exist $\varepsilon > 0$, a disjoint sequence $\{E_m\}_{m=1}^{\infty}$ in Σ, and a sequence $\{\nu_m\}_{m=1}^{\infty}$ in K such that

$$(*) \qquad \|\nu_m(E_m)\| > \varepsilon, \quad \forall_m.$$

By 5.1.9, we may assume that every μ_n is countably additive on the σ-algebra $\Sigma_0 = \Sigma(\{E_{n_k}\})$. By 5.1.7, $\{\mu_n|\Sigma_0\}_{n=1}^{\infty}$ is uniformly countably additive on Σ_0, contradicting (*). Hence K must be uniformly s-additive on Σ.

5.1.12 COROLLARY. <u>If</u> λ <u>and</u> λ_n <u>are in</u> $ba(\Sigma)$ $(n=1,2,\ldots)$, <u>then</u> $\lambda_n \to \lambda$ <u>weakly in</u> $ba(\Sigma)$ <u>if and only if</u> $\lambda_n(E) \to \lambda(E)$ <u>for every</u> E <u>in</u> Σ.

PROOF. $(\Rightarrow)$ clearly.

$(\Leftarrow)$. If $\lambda_n(E) \to \lambda(E)$ for every E in Σ, then by 5.1.10, $\{\lambda_n\}$ is uniformly bounded on Σ, and by 5.1.11, $\{\lambda_n\}$ is uniformly s-additive. Hence by 4.4.4, $\{\lambda_n\}$ is conditionally weakly compact. By the Eberlein-Smulian theorem, every subsequence has a subsequence converging weakly, and the limit in each case must be λ.

5.1.13 THEOREM. *Let* λ *be in* $ba^{+}(\Sigma)$, *and for every* n *let* $\mu_n : \Sigma \to X$ *be finitely additive and absolutely continuous with respect to* λ. *If for every* E *in* Σ,

$$\mu(E) = \lim_{n\to\infty} \mu_n(E)$$

exists, then $\{\mu_n\}$ *is uniformly absolutely continuous with respect to* λ.

PROOF. By 5.1.5, each μ_n is s-additive. By 5.1.10, $\{\mu_n\}$ is uniformly bounded, and by 5.1.11, $\{\mu_n\}$ is uniformly s-additive. Hence $K = \{x^* \circ \mu_n : n = 1, 2, \ldots; \; x^* \in X^*, \|x^*\| \leq 1\}$ is uniformly bounded and uniformly s-additive. Thus K is conditionally weakly compact by 4.4.4. Since λ is a control measure for K, it is a uniform control measure for K by 4.4.7.

5.1.14 LEMMA. *If* K *is a bounded subset of* $ba(\Sigma)$ *which is not uniformly s-additive, then there exists* $\delta > 0$ *such that for every* $\varepsilon > 0$ *there is a sequence* $\{\mu_n\}$ *in* K *and a disjoint sequence* $\{E_n\}$ *in* Σ *such that for every* n

(i) $\quad |\mu_n(E_n)| > \delta$

and (ii) $|\mu_n|(\bigcup_{j\neq n} E_j) < \varepsilon$.

PROOF. Since K is not uniformly s-additive on Σ, there exists $\delta_1 > 0$, a disjoint sequence $\{A_n\}$ in Σ, and a sequence $\{\lambda_n\}$ in K such that

$$(*) \qquad |\lambda_n(A_n)| > \delta_1, \quad \forall_n.$$

By 5.1.9, we may assume without loss of generality that every $|\lambda_n|$ is countably additive on the σ-algebra $\Sigma_0 = \Sigma(\{A_n\}_{n=1}^{\infty})$. By (*), $K_0 = \{\lambda_n|\Sigma_0\}_{n=1}^{\infty}$ is not uniformly s-additive on Σ_0. By 4.2.10, there exists $\delta > 0$ such that for every $\varepsilon > 0$ there is a sequence $\{\mu_n\}$ in K_0, and a disjoint sequence E_n in $\Sigma_0 \subset \Sigma$, such that for every n

$$|\mu_n(E_n)| > \delta$$

and

$$|\mu_n|(\bigcup_{j\neq n} E_j) = \sum_{j\neq n} |\mu_n|(E_j) < \varepsilon.$$

5.1.15 THEOREM. Let $\nu:\Sigma \to X$ be a bounded finitely additive measure. If ν is not s-additive, then there exists an isomorphism φ on the space ℓ^{∞} into X. In fact, we can choose a disjoint sequence $\{E_n\}$ in Σ such that $\varphi(\delta_n) = \nu(E_n)$, where $\delta_n \in \ell^{\infty}$ is the sequence given by $\delta_n(m) = \delta_{mn}$.

PROOF. If ν is not s-additive, then the set $\{x^* \circ \nu : x^* \in X^*, \|x^*\| \leq 1\}$ is not uniformly s-additive. Since ν is bounded, there exists M such that $|x^* \circ \nu|(\Omega) \leq M$ for all $\|x^*\| \leq 1$. By 5.1.14, there exists $\delta > 0$, a disjoint sequence $\{E_n\}$ in Σ, and a sequence $\{x_n^*\}$ in the unit ball of X^* such that

$$|x_n^* \circ \nu(E_n)| > \delta$$

and

$$|x_n^* \circ \nu|(\bigcup_{j \neq n} E_j) < \frac{\delta}{2}.$$

Let ℓ_0^∞ denote the dense subspace of ℓ^∞ consisting of all finite-valued sequences

$$\alpha = \sum_{i=1}^{n} \beta_i \chi_{A_i},$$

where $\{A_1, \ldots, A_n\}$ is a partition of $\mathbb{N}$. Define $\varphi_0(\alpha)$ by

$$\varphi_0(\alpha) = \sum_{i=1}^{n} \beta_i \nu(\bigcup_{j \in A_i} E_j).$$

Then φ_0 is linear. For $x^* \in X^*$, $\|x^*\| \leq 1$, we have

$$|x^*(\varphi_0(\alpha))| = \sum_{i=1}^{n} |\beta_i| |x^* \circ \nu| (\bigcup_{j \in A_i} E_j)$$

$$\le \|\alpha\|_\infty |x^* \circ \nu| (\Omega) \le M \|\alpha\|_\infty ,$$

so that φ_0 is continuous.

Now given $m \in N$, choose A_{i_m} such that $m \in A_{i_m}$. Then

$$|x_m^*(\varphi_0(\alpha_0))| = |\sum_{i=1}^{n} \beta_i x_m^* \circ \nu(\bigcup_{j \in A_i} E_j)|$$

$$\ge |\beta_{i_m} x_m^* \circ \nu(E_m)| - |\sum_{i=1}^{n} \beta_i x_m^* \circ \nu (\bigcup_{\substack{j \in A_i \\ j \ne m}} E_j)|$$

$$\ge |\beta_{i_m}| |x_m^* \circ \nu(E_m)| - \|\alpha\|_\infty \sum_{i=1}^{n} |x_m^* \circ \nu| (\bigcup_{\substack{j \in A_i \\ j \ne m}} E_j)$$

$$\ge |\beta_{i_m}| \delta - \|\alpha\|_\infty |x_m^* \circ \nu| (\bigcup_{j \ne m} E_j)$$

$$\ge |\beta_{i_m}| \delta - \|\alpha\|_\infty \cdot \frac{\delta}{2} .$$

Taking the supremum over m gives

$$\|\varphi_0(\alpha)\| \ge \sup_m |x_m^* \circ \varphi_0(\alpha)| \ge \|\alpha\|_\infty \cdot \frac{\delta}{2} ,$$

so that φ_0 is one-to-one and φ_0^{-1} is continuous.

Since ℓ_0^∞ is dense in ℓ^∞, φ_0 extends to an isomorphism of ℓ^∞ into X.

5.1.16 CORollary. *For a Banach space* X, *the following two statements are equivalent.*

(1) *For every* σ-*algebra* Σ, *every bounded finitely additive measure* $\mu:\Sigma \to X$ *is* s-*additive.*

(2) X *does not contain an isomorphic copy of* ℓ^∞.

PROOF. (2) $\Rightarrow$ (1) by the theorem. To see that (1) $\Rightarrow$ (2), note that $\nu:2^N \to \ell^\infty$ given by $\nu(A) = \chi_A$ $(A \subset N)$ is a bounded finitely additive measure which is not s-additive.

5.1.17 COROLLARY. *If* X *is* separable, *then for every* σ-*algebra* Σ, *every bounded finitely additive measure* $\mu:\Sigma \to X$ *is* s-*additive.*

5.2 THE PETTIS-ORLICZ THEOREM

5.2.1 A measure $\mu:\Sigma \to X$ is said to be <u>weakly countably additive</u> if $x^* \circ \mu$ is countably additive for all $x^* \in X^*$. It follows from the uniform boundedness principle that such a μ is bounded. Also, if μ is such a measure and $\{E_n\}$ is a disjoint sequence in Σ, then

$$x^*[(\bigcup_{n=1}^{\infty} E_n)] = \sum_{n=1}^{\infty} x^*\mu(E_n) = \lim_N x^* \sum_{n=1}^{N} \mu(E_n),$$

so that $\mu(\bigcup_{n=1}^{\infty} E_n) = wk - \sum_{n=1}^{\infty} \mu(E_n)$; i.e., μ is countably additive into the weak topology. Clearly if μ is s-additive and weakly countably additive, then it is countably additive.

In this section we prove the powerful theorem of Pettis which says that if μ is weakly countably additive then it is countably additive. (Actually, this follows immediately from 4.2.14, but we give an alternate proof here.)

5.2.2 LEMMA. <u>A sequence $\{\alpha^{(n)}\}_{n=1}^{\infty}$ in ℓ^1 converges in norm to zero if and only if for all β in ℓ^∞ with $\beta_i = 0, 1,$ or -1, $\forall i$,</u>

$$\sum_{i=1}^{\infty} \beta_i \alpha_i^{(n)} \to 0$$

<u>as $n \to \infty$. In particular, for sequences in ℓ^1, weak and norm convergence agree.</u>

PROOF. ($\Rightarrow$) clear.

($\Leftarrow$). Suppose $\{\alpha^{(n)}\}$ does not converge in norm to zero. By passing to a subsequence we may assume there exists $\varepsilon > 0$ such that $\|\alpha^{(n)}\|_1 \geq \varepsilon$ for all n. Again by passing to a subsequence (using the Cantor diagonal method) we can assume

$$\operatorname{sgn} \alpha_i^{(n)} = \operatorname{sgn} \alpha_i^{(m)}, \quad \forall i,m,n.$$

Let $\beta_i = \operatorname{sgn} \alpha_i^{(n)}$ for all i. Then

$$\varepsilon \leq \|\alpha^{(n)}\|_1 = \sum_{i=1}^{\infty} |\alpha_i^{(n)}| = \sum_{i=1}^{\infty} \beta_i \alpha_i^{(n)} \to 0,$$

a contradiction.

5.2.3 LEMMA. <u>If $\mu: 2^{\mathbb{N}} \to X$ is weakly countably additive, then μ is countably additive.</u>

PROOF. For each n, let $x_n = \mu(\{n\})$, and let Y denote the closed linear span of $\{x_n\}_{n=1}^{\infty}$. If E is a subset of $\mathbb{N}$, then $\mu(E) = \text{wk} - \sum_{n \in E} x_n$ is in Y. Hence we may as well assume $X = Y$, and thus that X is separable. By , the unit ball of X^* is metrizable in the weak* topology.

Let $\mu^*: X^* \to ca(2^{\mathbb{N}})$ be the natural "adjoint" of μ. We show that μ^* is weak*-weak continuous so that by 5.1.5, μ is s-additive and hence countably additive.

Note that $\psi : ca(2^{\mathbb{N}}) \to \ell^1$ given by

$$\psi(\nu) = \{\nu(\{n\})\}_{n=1}^{\infty}$$

is an isometric isomorphism. We show $\psi \circ \mu^*$ is weak*-norm continuous on the unit ball of X^*. By , this implies $\psi \circ \mu^*$ is weak*-weak continuous on all of X^*. For $x^* \in X^*$ we have

$$\psi \circ \mu^*(x^*) = \psi(x^* \circ \mu) = \{x^*(x_i)\}_{i=1}^{\infty}.$$

Suppose $\{x_m^*\}$ is in the unit ball of X^* and $x_m^* \to x^*$ weak*. We are to show $\|\{(x_m^* - x^*)(x_i)\}_{i=1}^{\infty}\|_1 \to 0$ as $m \to \infty$. Given β in ℓ^∞ with $\beta_i = 1$, or -1, $\forall i$, we have

$$\sum_i \beta_i [(x_m^* - x^*)(x_i)] = \sum_{\beta_i = 1} (x_m^* - x^*)(x_i) - \sum_{\beta_i = -1} (x_m^* - x^*)(x_i)$$

$$= (x_m^* - x^*)[(wk - \sum_{\beta_i = 1} x_i) - (wk \sum_{\beta_i = -1} x_i)]$$

$$\to 0.$$

By the previous lemma, $\|\{(x_m^* - x^*)(x_i)\}_{i=1}^{\infty}\|_1 \to 0$.

5.2.4 THEOREM (Pettis). If Σ is a σ-algebra and $\mu:\Sigma \to X$ is weakly countably additive, then μ is countably additive.

PROOF. Let $\{E_n\}_{n=1}^{\infty}$ be a disjoint sequence in Σ and define $\nu:2^{\mathbb{N}} \to X$ by

$$\nu(A) = wk - \sum_{n\in A} \mu(E_n) = \mu(\bigcup_{n\in A} E_n), \quad (A \subset \mathbb{N}).$$

Then ν is weakly countably additive and hence countably additive. Hence

$$\nu(\mathbb{N}) = \text{norm} - \sum_{n=1}^{\infty} \nu(\{n\}) = \text{norm} - \sum_{n=1}^{\infty} \mu(E_n).$$

5.2.5 COROLLARY (Pettis-Orlicz). Let Σx_n be a series in a Banach space X such that each subseries converges weakly in X. Then Σx_n converges unconditionally in norm.

PROOF. Exercise.

5.2.6 COROLLARY (Extension Theorem). Let A be an algebra, and let $\mu:A \to X$ be countably additive on A. Then μ has an extension to a countably additive measure $\bar{\mu}:\Sigma(A) \to X$ if and only if μ is s-additive. Moreover, in that case, for any E in Σ,

$$\overline{\mu}(E) = wk - \lim_{\pi} \sum_{n=1}^{\infty} \mu(A_n),$$

<u>where the limit is taken over</u> $\pi \in P_A(E)$ (see 1.1.6).

PROOF. ($\Leftarrow$) since $\overline{\mu}$ must be s-additive.

($\Rightarrow$). Let E be in Σ. For every $\pi = \{A_n\}_{n=1}^{\infty}$ in $P_A(E)$, let $x_\pi = \sum_{n=1}^{\infty} \mu(A_n)$. Since $\mu(A)$ is conditionally weakly compact, some subset of $\{x_\pi\}$ converges weakly in X. By 4.1.4, for every x^* in X^*, $\{x^*(x_\pi)\}_\pi$ converges, and hence $\{x_\pi\}$ itself converges weakly to some element which we denote by $\overline{\mu}(E)$. Now by 4.1.4, $x^* \circ \overline{\mu}$ is the unique extension of $x^* \circ \mu$ to a countably additive measure on Σ to $\mathbb{R}$; in particular, $\overline{\mu}$ is weakly countably additive, hence countably additive.

5.3. THE YOSIDA-HEWITT DECOMPOSITION THEOREM.

Throughout, X is a Banach space.

5.3.1. LEMMA. Suppose $f_n : S \to X$ has conditionally weakly compact range for all n and that $f_n \to f$ uniformly on S. Then f has conditionally weakly compact range.

PROOF. Let $f^* : X^* \to m(S)$ and $f^{**} : m(S)^* \to X^{**}$ be as in §5.1, and similarly define f_n^* and f_n^{**} for all n. We have

$$\|f_n^{**} - f^{**}\| = \|f_n^* - f^*\| = \|f_n - f\|_\infty \to 0,$$

so in particular $f_n^{**}(\gamma) \to f^{**}(\gamma)$ is norm for all γ in $m(S)^*$. Since each f_n^{**} takes its values in X, so does f^{**} and the lemma follows from 5.1.2.

5.3.2. If $\mathfrak{a}$ is any algebra of sets, let $sba(\mathfrak{a},X)$ denote the space of all s-additive measures $\mu : \mathfrak{a} \to X$, with the uniform norm. It follows from 5.3.1. that $sba(\mathfrak{a},X)$ is a closed subspace of $ba(\mathfrak{a},X)$, hence is a Banach space.

5.3.3. Let $\mathcal{P}$ denote the collection of all countable $\mathfrak{a}$-partitions $\pi = \{E_n\}_{n=1}^{\infty}$ of Ω. If $\pi = \{E_n\}$ is in $\mathcal{P}$, if μ is in $sba(\mathfrak{a},X)$, and if E is in $\mathfrak{a}$, let

$$t_\pi \mu(E) = \sum_{n=1}^{\infty} \mu(E \cap E_n).$$

Then $t_\pi \mu$ is finitely additive and has its range in the closure of the range of μ. Thus $t_\pi \mu$ is again in $sba(G,X)$ and $\|t_\pi \mu\|_\infty \leq \|\mu\|_\infty$. It follows that $t_\pi : sba(G,X) \to sba(G,X)$ is continuous and linear with norm one.

By definition, if π_1 and π_2 are in P, then the composition $t_{\pi_1} \circ t_{\pi_2}$ is equal to t_{π_3}, where π_3 is the least common refinement of the partitions π_1 and π_2. Note that a member μ of $sba(G,X)$ is countably additive if and only if $t_\pi \mu = \mu$ for all π in P. Thus $T = \{t_\pi : \pi \in P\}$ is a commutative semi-group of idempotent operators whose set of common fixed points is the set of countably additive members of $sba(G,X)$.

5.3.4. LEMMA. Let S be a semigroup with a compact topology such that multiplication is separately continuous (i.e., for fixed s_0 in S the functions $s \mapsto s_0 s$ and $s \mapsto s s_0$ are each continuous). Let T be a dense, commutative sub-semi-group of idempotent elements. Then T is directed by the partial ordering $\geq$ defined by

$$s_1 \geq s_2 \Leftrightarrow s_1 s_2 = s_1.$$

Moreover, T, considered as a net in S, converges to a zero θ for S; i.e., θ satisfies

$$(\#) \qquad \theta s = s\theta = \theta$$

for all s in S.

PROOF. That $\geqq$ directs T is trivial. Since there is at most one element θ in S which satisfies (#), to complete the proof it suffices to show that the limit of any convergent subset of T must satisfy (#). Thus, let $\{t_\alpha\}_\alpha$ be any subset of T which converges to an element θ. If t is in T, then $\{t_\alpha\}_\alpha$ is eventually greater than or equal to t, so that

$$t\theta = \lim_\alpha tt_\alpha = \lim_\alpha t_\alpha = \theta$$

By density of T, θ satisfies (#).

5.3.5. THEOREM. *Let Y be a Banach space, let T be a commutative semigroup of idempotent linear operators on Y, and suppose that for every element y in Y the set*

$$O(y) = \text{weak closure of } \{t(y): t \in T\}$$

is weakly compact in Y. Then the following statements hold.

(i) For every y in Y there is one and only one T-fixed point $\theta(y)$ in $O(y)$; i.e., one point $\theta(y)$ such that $t(\theta(y)) = \theta(y)$ for all t in T.

(ii) The mapping θ: $Y \to Y$ given by (i) is continuous and linear. Moreover, T, directed as in the lemma, converges pointwise to θ; i.e., for every y in Y

$$\lim_{t \in T} \|\theta(y) - t(y)\| = 0$$

(iii) The space Y can be written as a direct sum

$$Y = Y_1 \oplus Y_0$$

where $Y_1 = \{y \in Y : y \text{ is } T\text{-fixed}\} = \theta(Y)$ and $Y_0 = \{y \in Y : 0 \in O(y)\} = \ker\theta$.

PROOF. Let $Q = \prod_{y \in Y} (O(y), \text{weak})$ have the product topology. Then Q is compact and contains T. Let S denote the closure of T in Q. By the uniform boundedness theorem, T is uniformly bounded and hence each member of S is a continuous linear operator on Y to Y. Suppose $s_\alpha \to s$ in S, and let s_0 be in S. Then for every y in Y,

$$s_\alpha \circ s_o(y) \to s \circ s_o(y)$$

and

$$s_o \circ s_\alpha(y) \to s_o \circ s(y)$$

weakly in Y, so $s_\alpha \circ s_o \to s \circ s_o$ and $s_o \circ s_\alpha \to s_o \circ s$ in Q. In particular, if $\{t_\alpha^{(1)}\}$ and $\{t_\beta^{(2)}\}$ are two nets in T converging to s_1 and s_2 in S, then

$$s_1 \circ s_2 = \lim_\alpha t_\alpha^{(1)} \circ s_2 = \lim_\alpha \lim_\beta t_\alpha^{(1)} \circ t_\beta^{(1)},$$

and so S is closed under compositions. Thus S and T satisfy the hypotheses of 5.3.4. Moreover, for every y in Y, $O(y) = \{s(y) : s \in S\}$.

By the lemma, T converges in S to some zero Θ for S. Since $t \circ \Theta = \Theta$ for all t in T, $\Theta(y)$ is a T-fixed point for every y in Y. If z is any T-fixed-point in $O(y)$, then $z = s(y)$ for some s in S and so

$$z = \Theta(z) = \Theta(s(y)) = \Theta(y).$$

Hence (i) holds, and Θ is continuous and linear.

Next, let y be in Y. Then $\Theta(y)$ is in the norm closed convex hull of $\{t(y) : t \in T\}$ (by .) Thus, given $\epsilon > 0$ there exist $t_1, \ldots, t_n$ in T and $\alpha_i \geq 0$

$(i = 1,\ldots,n)$ with $\sum_{i=1}^{n} \alpha_i = 1$ and $\|\theta(y) - \sum \alpha_i t_i(y)\| < \frac{\epsilon}{m+1}$,

where $M = \sup\{\|t\| : t \in T\}$. Then $t \geqq t_1 \circ \ldots \circ t_n$ implies

$$\|\theta(y) - t(y)\| = \| t[\theta(y) - \sum \alpha_i t_i(y)]\|$$

$$\leqq M \frac{\epsilon}{M+1} < \epsilon,$$

so $\lim_t \|\theta(y) - t(y)\| = 0$. This establishes (ii).

If y is in Y, then

$$y = \theta(y) + (y - \theta(y)),$$

$\theta(y)$ is in

$$Y_1 = \{y_1 \in Y : y_1 \text{ is } T\text{-fixed}\} = \theta(Y),$$

and $(y - \theta(y))$ is in

$$Y_o = \ker\theta = \{y_o \in Y : 0 \in O(y_o)\}.$$

Since $Y_1 \cap Y_o = \{0\}$, this establishes (iii).

5.3.6. LEMMA. Let μ be in $sba(\mathfrak{a},X)$, and for every E in $\mathfrak{a}$, let $\mu_E : \mathfrak{a} \to X$ be defined by

$$\mu_E(F) = \mu(E \cap F) \quad , (F \in \mathfrak{a}).$$

Then $\{\mu_E : E \in \mathfrak{a}\}$ is a conditionally weakly compact subset of $sba(\mathfrak{a}, X)$.

5.3.6.

PROOF. Choose λ in $ba(\mathcal{G})$ such that

$$\lim_{|\lambda|(E)\to 0} |x^* \circ \mu|(E) = 0$$

uniformly over $\|x^*\| \leq$. Let $U: \mathcal{G} \to sba(\mathcal{G},X)$ be given by $U(E) = \mu_E$. Then U is finitely additive. We show U is s-additive which will complete the proof (by 5.15.) Suppose $\{E_n\}_{n=1}^{\infty}$ is a disjoint sequence in $\mathcal{G}$. Then $|\lambda|(E_n) \to 0$, so

$$\begin{aligned} \|U(E_n)\|_\infty &= \|\mu E_n\|_\infty \\ &= \sup_{F\in\mathcal{G}} \|\mu(E_n \cap F)\| \\ &= \sup_{\|x^*\|\leq 1} \sup_{F\in\mathcal{G}} |x^*\mu(E_n \cap F)| \\ &\leq \sup_{\|x^*\|\leq 1} |x^*\mu|(E_n) \\ &\to 0. \end{aligned}$$

5.3.7. COROLLARY. <u>If</u> μ <u>is in</u> $sba(\mathcal{G})$, <u>then</u>

$$\{t_\pi \mu \ : \ \pi \in \mathcal{P}\}$$

<u>is conditionally weakly compact in</u> $sba(\mathcal{G})$.

PROOF. If $\pi = \{E_n\}_1^\infty$ is in $\mathcal{P}$, then

$$t_\pi\mu(E) = \lim_n \mu\left(\bigcup_{i=1}^n E_i\right)(E) \quad , \forall E\in \mathcal{G}.$$

Now $\{\mu(\bigcup_{i=1}^{n} E_i)\}_{n=1}^{\infty}$ has a subsequence which converges weakly (by 5.3.6 and the Eberlein-Smulian theorem) and the limit must be $t_\pi\mu$. Hence $t_\pi\mu$ is in the weak closure of $\{\mu_E : E \in \mathcal{G}\}$.

5.3.8. A member μ of $sba(\mathcal{G},X)$ is <u>purely finitely additive</u> if and only if for every $\epsilon > 0$ there exists a partition $\pi = (E_n)_{n=1}^{\infty}$ in $\mathcal{P}$ such that $\|t_\pi\mu\|_\infty < \epsilon$; i.e.,

$$\left\|\sum_{n=1}^{\infty} \mu(E \cap E_n)\right\| < \epsilon \quad , \forall E \in \mathcal{G}.$$

We let $pba(\mathcal{G},X)$ denote the purely finitely additive members of $sba(\mathcal{G},X)$.

Moreover, let $sca(\mathcal{G},X)$ denote the space of countably additive members of $sba(\mathcal{G},X)$.

5.3.9. THEOREM. (The Yosida-Hewitt Decomposition Theorem). <u>For any algebra</u> $\mathcal{G}$ <u>and any Banach space</u> X, <u>the following statements hold.</u>

(1) $pba(\mathcal{G},X)$ <u>is a closed subspace of</u> $sba(\mathcal{G},X)$.

(2) $sba(\mathcal{G},X) = sca(\mathcal{G},X) \oplus pba(\mathcal{G},X)$; <u>i.e., every</u> μ <u>in</u> $sba(\mathcal{G},X)$ <u>has a unique decomposition</u> $\mu = \mu_c + \mu_p$, <u>where</u> $\mu_c \in sca(\mathcal{G},X)$, $\mu_p \in pba(\mathcal{G},X)$.

(3) <u>The projection</u> θ: $\mu \to \mu_c$ <u>is continuous and linear of norm one.</u>

(4) *For every* μ *in* $sba(G)$,

$$\Theta(\mu) = \lim_{\pi \in p} t_\pi(\mu)$$

(norm limit).

PROOF. The statements follows from 5.3.5.

5.3.10. PROPOSITION. (i) *If* μ *is in* $ba(G)$ *and* $\mu \geqq 0$, *then* μ *is purely finitely additive if and only if*

$$\nu \in ca(G) \quad , \quad 0 \leqq \nu \leqq \mu \Rightarrow \nu = 0.$$

(ii) *If* μ *is in* $ba(G)$, *then* μ *is purely finitely additive if and only if* μ^+ *and* μ^- *both are.*

(iii) *If* μ *is in* $ba(G)$, *then* μ *is purely finitely additive if and only if* $x^* \circ \mu$ *is purely finitely additive for every* x^* *in* X^*.

PROOF. (i). If $\Theta : ba(G) \to ca(G)$ is the map given by the theorem when $X = \mathbb{R}$, then for $\mu \geqq 0$

$$0 \leqq \Theta(\mu) \leqq \mu$$

since $0 \leqq t_\pi \mu \leqq \mu$ for all π. In particular, Θ is order preserving.

Thus if μ satisfies the condition in (i), then $\Theta(\mu) = 0$ and μ is purely finitely additive. Conversely, if μ

is purely finitely additive, $\nu \in ca(G)$, and $0 \leqq \nu \leqq \mu$ imply

$$0 \leqq \nu = \Theta(\nu) \leqq \Theta(\mu) = 0,$$

so $\nu = 0$.

(ii). If μ^+ and μ^- are purely finitely additive then $\Theta(\mu) = \Theta(\mu^+) - \Theta(\mu^-) = 0$.

Conversely, suppose $\Theta(\mu) = 0$. Then

$$\mu = \mu - \Theta(\mu) = [\mu^+ - \Theta(\mu^+)] - [\mu^- - \Theta(\mu^-)].$$

Since $\mu^+ - \Theta(\mu^+) \geqq 0$ and $\mu^- - \Theta(\mu^-) \geqq 0$, we have $\mu^+ - \Theta(\mu^+) \geqq \mu^+$ and $\mu^- - \Theta(\mu^-) \geqq \mu^-$, and it follows that $\Theta(\mu^+) = 0$ and $\Theta(\mu^-) = 0$.

(iii) If μ is in $pba(G,X)$, then by definition $x^* \circ \mu$ is purely finitely additive for any x^* in X^*. Conversely, suppose $x^* \circ \mu$ is purely finitely additive for all $x^* \in X^*$. If $\Theta\mu \neq 0$, then for some E in G and x^* in X^*, $0 \neq x^*\Theta\mu(E) = \lim_{\pi} x^* t_{\pi}\mu(E) = \lim_{\pi} \sum_{n=1}^{\infty} x^*\mu(E \cap E_n) = 0$ a contradiction.

R.5.1. REMARKS AND REFERENCES.

(1) Theorem 5.1.5 is a combination of results of Dunford and Schwartz [1958, p. 314], Uhl [1971], and Brooks [1971]. (The last statement of 5.1.5 is from Huff and Morris [1973].) Corollary 5.1.6(a),(b),(c) were proved earlier by Bartle, Dunford, and Schwartz [1955] as well as the existence of a control measure for μ ; the full strength of 5.1.6 (d) follows from the results of §3.3 (see R.3.3 for references). Finally, 5.1.6 (e) was first proved by Pettis [1938][1939].

(2) The important Theorem 5.1.9 is due to Drewnowski [1972, Prop.1].

(3) Theorem 5.1.10 which generalizes Nikodym's theorem 3.2.10 was proved by Darst [1973].

(4) Theorems 5.1.11 and 5.1.13 which generalizes Nikodym's theorem 3.2.9 and the Vitali-Hahn-Saks theorem 3.2.8 are due to Andô [196[illegible]] for the scalar-valued case, and to Brooks and Jewett [1970] for the general case.

(5) Lemma 5.1.14 is due to Rosenthal [1968], Lemma 1]. Theorem 5.1.15 (5.1.16) is due to Diestel and Faires [1973]; the proof here is from Uhl [1973]. Corollary 5.1.17 had been proved earlier by Diestel [1973(a)]. These results should be compared with 4.2.9-4.2.13.

<u>Additional</u> <u>References:</u> Rosenthal [1970], Brooks [1973], Diestel [1973(b)], Gould [1965], Hoffmann-Jørgensen [1971], Tweddle [1970], Labuda [1972], Drewnowski [1972(a)].

R.5.2. REMARKS AND REFERENCES.

(1) The main results here (5.2.4, 5.2.5) are due to Orlicz and Pettis, see Pettis [1938] [1939].

(2) Lemma 5.2.2 goes back at least to Banach's book [1932, p. 123].

(3) Theorem 5.2.6 is from Uhl [1971]. See also Kluvanek [1973].

Additional References: (Pettis-Orlicz theorem) Bessaga and Pelczynski [1958], Kalton [1971], McArthur [1967], Grothendieck [1953], Tweddle [1970]; (The Extension theorem) Fox [1968], Dinculeanu and Kluvanek [1967].

R.5.3. REMARKS AND REFERENCES.

Yosida and Hewitt [1952], using 5.3.10 (i) and 5.3.10 (ii) as the definition of purely finitely additive, proved the decomposition theorem for the scalar case. Using the Yosida-Hewitt result and using 5.3.10 (iii) as definition, Uhl [1971] proved the theorem for the vector-valued case The proof given here of the general case is due to Huff [1973 (a)].

The general ergodic theorem 5.3.5 is a special case of known ergodic theorems (see especially Barry [1954]).

Additional References: Andô [1961], Brooks [1969], Chatterji [1968].