

#### 4. THE SPACE $ba(\mathcal{A})$

##### 4.1 VARIATION

Throughout this chapter, let  $\mathcal{A}$  be an algebra of subsets of a non-void set  $\Omega$ . We consider the space  $ba(\mathcal{A})$  of all bounded finitely additive functions  $\mu: \mathcal{A} \rightarrow \mathbb{R}$ .

For finitely additive measures which are not countably additive, the Hahn decomposition theorem fails. We resort to the remarks in 3.1.6 in order to define the positive and negative parts and the variation of a finitely additive measure.

4.1.1 THEOREM.  $ba(\mathcal{A})$  is a vector lattice under the usual ordering with

$$\mu^+(E) = \sup\{\mu(F): F \subset E, F \in \Sigma\}$$

$$\mu^-(E) = -\inf\{\mu(F): F \subset E, F \in \Sigma\}$$

and  $|\mu|(E) = \mu^+(E) + \mu^-(E)$

$$= \sup_{\pi} \sum_{i=1}^n |\mu(E \cap E_i)|$$

where the last supremum is taken over all finite partitions

$$\pi = \{E_1, \dots, E_n\} \text{ of } \Omega.$$

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PROOF. Let  $\mu^+$  be defined as above. If  $E_1, E_2$  are in  $\Sigma$  and  $E_1 \cap E_2 = \emptyset$ , then

$$\begin{aligned}\mu^+(E_1 \cup E_2) &= \sup\{\mu(F \cap E_1) + \mu(F \cap E_2) : F \in A, F \subset E_1 \cup E_2\} \\ &= \sup\{\mu(F_1) + \mu(F_2) : F_1, F_2 \in A, F_1 \subset E_1, F_2 \subset E_2\} \\ &= \mu^+(E_1) + \mu^+(E_2).\end{aligned}$$

Thus  $\mu^+$  is in  $\text{ba}(\Sigma)$ . Clearly  $\mu^+ \geq 0$  and  $\mu^+ \geq \mu$ . If  $v \geq 0$  and  $v \geq \mu$ , then

$$F \subset E \Rightarrow \mu(F) \leq v(F) \leq v(E),$$

so  $\mu^+ \leq v$ . Therefore  $\text{ba}(A)$  is indeed a vector lattice with  $\mu \vee 0 = \mu^+$ .

We have (as in any vector lattice)  $\mu^- = (-\mu)^+$ , and so

$$\begin{aligned}\mu^-(E) &= \sup\{-\mu(F) : F \in \Sigma, F \subset E\} \\ &= -\inf\{\mu(F) : F \in \Sigma, F \subset E\}.\end{aligned}$$

Also (as in any vector lattice),  $|\mu| = \mu^+ + \mu^-$ . For any  $E$  in  $\Sigma$  and  $\pi = \{E_1, \dots, E_n\}$  a partition of  $E$ ,

(el. l.)

$$\begin{aligned}
 2|\mu(E \cap E_j)| &= z^+ \mu(E \cap E_j) - z^- \mu(E \cap E_j) \\
 &= \mu((U^+ E_j) \cap E) - \mu((U^- E_j) \cap E) \\
 &\leq \mu^+(E) + \mu^-(E) \\
 &= |\mu|(E),
 \end{aligned}$$

where  $z^+(z^-)$  means the sum over only those for which  $\mu(E \cap E_j) \geq 0 (< 0)$  and similarly for  $U^+$  and  $U^-$ . Finally, if  $F \subseteq E$ , then

$$\begin{aligned}
 2\mu(F) &= \mu(E) + \mu(F) - \mu(E \setminus F) \\
 &\leq \mu(E) + \sup_{E_j} z|\mu(E \cap E_j)|.
 \end{aligned}$$

so

$$\begin{aligned}
 2\mu^+(E) &\leq \mu(E) + \sup_{E_j} z|\mu(E \cap E_j)| \\
 &= \mu^+(E) + \mu^-(E) + \sup_{E_j} z|\mu(E \cap E_j)|,
 \end{aligned}$$

and therefore  $|\mu|(E) = \mu^+(E) + \mu^-(E) \leq \sup_{E_j} z|\mu(E \cap E_j)|$ .

4.1.2 Let  $S(A)$  denote the linear space of all  $A$ -simple functions

$$f = \sum_{j=1}^n c_j X_{A_j} \quad (c_j \in \mathbb{R}, A_j \in A). \text{ Then } S(A) \text{ is a normed linear}$$

space with the uniform norm  $\|\cdot\|_\infty$ . If  $\mu$  in  $\text{ba}(A)$ , define  $\varphi_\mu : S(A) \rightarrow \mathbb{R}$  by  $\varphi_\mu(f) = \sum_{i=1}^n a_i \mu(A_i)$ . It is easily checked that  $\varphi_\mu$  is a linear functional on  $S(A)$  which is continuous and its operator norm is given by  $\|\varphi_\mu\| = |\mu|(A)$ . Since  $\mu \mapsto \varphi_\mu$  is one-to-one and linear, we have that the quantity

$$\|\mu\| = |\mu|(A)$$

(called the total variation norm of  $\mu$ ) defines a norm on  $\text{ba}(A)$ . Since conversely, every  $\varphi$  in  $S(A)^*$  is equal to  $\varphi_\mu$  where  $\mu(A) = \varphi(X_A)$ , we have that  $(\text{ba}(A), \|\cdot\|)$  is a Banach space isometrically isomorphic to  $(S(A), \|\cdot\|)^*$ .

From 4.1.1, for any  $\mu$  in  $\text{ba}(A)$  and any  $E$  in  $A$ ,

$$|\mu(E)| \leq |\mu|(E) = \mu^+(E) + \mu^-(E) \leq 2\|\mu\|_\infty.$$

so  $\|\mu\|_\infty \leq \|\mu\| \leq 2\|\mu\|_\infty$ . Hence  $\|\cdot\|$  and  $\|\cdot\|_\infty$  are equivalent norms on  $\text{ba}(A)$ .

4.1.3 LEMMA. If  $\mu \in \text{ba}(A)$  is countably additive on  $A$ , then  $\mu^+$ ,  $\mu^-$  and  $|\mu|$  are countably additive on  $A$ .

PROOF. Suppose  $\{E_n\}_{n=1}^{\infty}$  is a disjoint sequence in  $A$  for which  $\bigcup_{n=1}^{\infty} E_n$  is in  $A$ . Since for every  $N$

$$\mu^+(\bigcup_{n=1}^N E_n) \geq \mu^+(\bigcup_{n=1}^N E_n) = \sum_{n=1}^N \mu^+(E_n),$$

$$\text{we have } \mu^+(\bigcup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{\infty} \mu^+(E_n).$$

Now given  $\epsilon > 0$ , choose  $F \in A$ ,  $F \subset \bigcup_{n=1}^{\infty} E_n$  such that

$$\mu^+(\bigcup_{n=1}^{\infty} E_n) < \mu(F) + \epsilon.$$

since  $\mu(F) = \sum_{n=1}^N \mu(F \cap E_n)$ , for some  $N$ ,

$$\mu(F) + \epsilon < \sum_{n=1}^N \mu(F \cap E_n) + 2\epsilon \leq \sum_{n=1}^{\infty} \mu^+(E_n) + 2\epsilon.$$

Since  $\epsilon > 0$  was arbitrary,

$$\mu^+(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^+(E_n).$$

Therefore,  $\mu^+$  is countably additive on  $A$ , and hence so

$$\text{are } \mu^- (\text{---} \mu^+ \text{---} \mu) \text{ and } |\mu| (\mu^+ + \mu^-).$$

**6.1.4 THEOREM (Extension Theorem).** If  $A$  is an algebra and  $Z = \Sigma(A)$ , then the restriction map  $r: ca(Z) \rightarrow ca(A)$  is an isometric isomorphism onto  $ca(A)$  which preserves the lattice operations. Moreover, if  $\mu$  is in  $ca(A)$ , then for every  $E$  in  $Z$ ,

$$[r^{-1}(\mu)](E) = \lim_{\alpha \rightarrow 1} \sum_{n=1}^{\infty} \mu(A_{n,\alpha}).$$

where the limit is taken over  $\alpha \in P_A(B)$  (see 1.1.6).

PROOF. Clearly  $r$  is linear. Suppose  $r(\mu) = r(\nu)$ . Let  $\lambda = |\mu| + |\nu|$ . Then  $A$  is  $\lambda$ -dense in  $E$  and both  $\mu$  and  $\nu$  are  $\lambda$ -continuous. Hence  $\mu \leq \nu$ , and  $r$  is one-to-one.

If  $\mu_0$  is in  $ca(A)$ , write  $\mu_0 = \mu_0^+ - \mu_0^-$ . Then  $\mu_0^+$  and  $\mu_0^-$  are in  $ca^+(A)$ , and hence by 1.2.4 they have extensions to  $\mu_1$  and  $\mu_2$  in  $ca^+(E)$ . Then  $r(\mu_1 - \mu_2) = \mu_0$ , so  $r$  is onto.

By 1.2.4, both  $r$  and  $r^{-1}$  are order preserving and hence  $r$  preserves the lattice operations.

The last statement of the theorem also follows from 1.2.4.

## 4.2 ABSOLUTE CONTINUITY - UNIFORM $\epsilon$ -ADDITIVITY

4.2.1 If  $\mu$  and  $\nu$  are in  $ba(A)$ , we say that  $\mu$  is absolutely continuous with respect to  $\nu$ , denoted  $\mu \ll \nu$ , provided for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$E \in A, |\nu|(E) < \delta \Rightarrow |\mu(E)| < \epsilon \text{ (and thus } |\mu|(E) \leq 2\epsilon).$$

This can be written  $\lim_{|\nu|(E) \rightarrow 0} \mu(E) = 0$ . If  $K$  is a subset of  $ba(A)$ , we say that  $K$  is uniformly absolutely continuous with respect to  $\lambda \in ba(A)$  if and only if

$$\lim_{|\lambda|(E) \rightarrow 0} \mu(E) = 0$$

uniformly over  $\mu$  in  $K$ .

4.2.2 Suppose  $\mu$  is in  $ba(A)$  and  $\{E_i\}$  is a disjoint sequence in  $A$ . Then

$$\sum_{i=1}^{\infty} |\mu(E_i)| \leq \sum_{i=1}^{\infty} |\mu|(E_i) \leq |\mu|(A) < \infty,$$

so  $\sum_{i=1}^{\infty} \mu(E_i)$  converges absolutely. We express this fact by saying that every  $\mu$  in  $ba(A)$  is strongly additive (or  $\sigma$ -additive). More generally, we say that a finitely additive function  $\mu: A \rightarrow X$  (where  $X$  is a Banach space) is  $\sigma$ -additive

if and only if for every disjoint sequence  $\{E_n\}_1^\infty$  in  $A$ ,

the series  $\sum_{n=1}^\infty \mu(E_n)$  converges unconditionally in  $X$ .

A family  $K$  of f.a. functions  $\mu:A \rightarrow X$  is said to be uniformly  $\sigma$ -additive if and only if for every disjoint sequence

$\{E_n\}_1^\infty$  in  $A$  the limit

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i)$$

exists uniformly over  $\mu$  in  $K$ ; i.e., for every  $\epsilon > 0$  there exists  $N$  such that

$$n \geq N \Rightarrow \left\| \sum_{i=1}^n \mu(E_i) - \sum_{i=1}^m \mu(E_i) \right\| < \epsilon, \quad \forall \mu \in K.$$

4.2.3 EXAMPLE. Let  $\mu:2^{\mathbb{N}} \rightarrow \ell^\infty$  be defined by  $\mu(A) = x_A$ , ( $A \subseteq \mathbb{N}$ ).

Then  $\mu$  is in  $ba(2^{\mathbb{N}}, \ell^\infty)$ , but it is not  $\sigma$ -additive.

4.2.4 PROPOSITION. If  $K$  is a family of finitely additive functions

$\mu:A \rightarrow X$ , then the following five statements are equivalent.

(1) If  $\{E_n\}$  is a disjoint sequence in  $A$ , then for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$n \geq n \geq N \Rightarrow \left\| \sum_{i=1}^n \mu(E_i) \right\| < \epsilon, \quad \forall \mu \in K.$$

(2)  $X$  is uniformly  $\sigma$ -additive on  $A$ .

(3) If  $\{E_\mu\}$  is a disjoint sequence in  $A$ , then for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$n \geq N \Rightarrow \|\mu(E_\mu)\| < \epsilon, \forall \mu \in X.$$

(4) If  $\{E_\mu\}$  is a monotone increasing sequence in  $A$ , then for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$n, n \geq N \Rightarrow \|\mu(E_\mu) - \mu(E_n)\| < \epsilon, \forall \mu \in X.$$

(5) If  $\{E_\mu\}$  is a monotone decreasing sequence in  $A$ , then for every  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$n, n \geq N \Rightarrow \|\mu(E_\mu) - \mu(E_n)\| < \epsilon, \forall \mu \in X.$$

If  $A$  is a  $\sigma$ -algebra and the members of  $X$  are countably additive, then  $X$  is uniformly  $\sigma$ -additive if and only if it is uniformly countably additive.

PROOF. As in 3.2.6.

4.2.5 COROLLARY. If  $\mu: A \rightarrow X$  is finitely additive, the following four statements are equivalent.

(1)  $\mu$  is  $\sigma$ -additive on  $A$ .

(2) If  $\{E_n\}$  is a disjoint sequence in  $A$ , then

$$\mu(E_n) \geq 0.$$

(3) If  $\{E_n\}$  is a monotone increasing sequence in  $A$ ,

then  $\lim_{n \rightarrow \infty} \mu(E_n)$  exists.

(4) If  $\{E_n\}$  is a monotone decreasing sequence in  $A$ ,

then  $\lim_{n \rightarrow \infty} \mu(E_n)$  exists.

4.2.6 COROLLARY. If  $K$  is a family of finitely additive functions

$\mu: A \rightarrow \mathbb{R}$ , and if  $K$  is uniformly absolutely continuous with

respect to some  $\lambda$  in  $ba(A)$ , then  $K$  is uniformly  $\sigma$ -additive.

PROOF. If  $\{E_n\}$  is a disjoint sequence in  $A$ , then  $|\lambda| (E_n) = 0$ , and so  $\mu(E_n) = 0$  uniformly over  $\mu$  in  $K$ .

4.2.7 COROLLARY. If  $\mu: A \rightarrow \mathbb{R}$  is finitely additive and  $\sigma$ -additive, then  $\mu$  is bounded.

PROOF. Define  $\bar{\mu}: A \rightarrow \mathbb{R} \cup \{-\infty\}$  by

$$\bar{\mu}(E) = \sup\{|\mu(F)| : F \in A, F \subseteq E\}.$$

It is easily checked that  $\bar{\mu}(E_1 \cup E_2) \leq \bar{\mu}(E_1) + \bar{\mu}(E_2)$ .

We claim that if  $E$  is in  $A$  and  $\bar{\mu}(E) = \infty$ , then given any  $M > 0$  there exists  $F$  in  $A$  such that

$$F \subset E, \quad \|\mu(F)\| > M, \quad \text{and} \quad \bar{\mu}(F) = \infty.$$

For, since  $\bar{\mu}(E) = \infty$  there exists  $A \in A$  with  $A \subset E$  and  $\|\mu(A)\| \geq \|\mu(E)\| + M$ . Then

$$\|\mu(E \setminus A)\| \geq \|\mu(E)\| - \|\mu(A)\| \geq M.$$

Since  $\infty = \bar{\mu}(E) \leq \bar{\mu}(A) + \bar{\mu}(E \setminus A)$ , we can take  $F$  to be one of  $A$  or  $(E \setminus A)$ .

Now if  $\mu$  is not bounded then  $\bar{\mu}(S) = \infty$ , and by induction we can find a decreasing sequence  $\{E_n\}$  in  $A$  such that  $\|\mu(E_n)\| \geq n$  for all  $n$ . This contradicts  $s$ -additivity ((4) of 4.2.5).

**4.2.8 PROPOSITION.** If  $K$  is a subset of  $ba(A)$ , then the following three statements are equivalent.

(1)  $K$  is uniformly  $s$ -additive on  $A$ .

(2)  $K^+ = \{\mu^+ : \mu \in K\}$  and  $K^- = \{\mu^- : \mu \in K\}$  are both uniformly  $s$ -additive on  $A$ .

(3)  $|K| = \{|\mu| : \mu \in K\}$  is uniformly  $s$ -additive on  $A$ .

PROOF. Using (3) of 4.2.4, clearly (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1). It remains to show (1)  $\Rightarrow$  (2).

If  $K^+$  is not uniformly  $s$ -additive, then there exists  $\epsilon > 0$ ,  $\{E_n\}$  a disjoint sequence in  $A$ , and  $\{\mu_n\} \subset K$  such that  $\mu_n^+(E_n) > \epsilon$  for all  $n$ . For each  $n$ , choose  $F_n \subset E_n$  such that  $\mu_n(F_n) > \epsilon/2$ . Thus  $K$  is not uniformly  $s$ -additive.

Similarly, if  $K^-$  is not uniformly  $s$ -additive, neither is  $K$ .

4.2.9 THEOREM. Let  $K$  be a bounded subset of  $ba(A)$ , and suppose  $K$  is not uniformly  $s$ -additive on  $A$ . Then there exists  $\delta > 0$  such that for every  $\epsilon > 0$  there exists a sequence  $\{\mu_n\}_{n=1}^\infty$  in  $K$ , and a disjoint sequence  $\{E_n\}$  in  $A$  such that for all  $n$ ,

$$(i) |\mu_n|(E_j) < \frac{\epsilon}{2^j} \text{ for all } j = 1, 2, \dots, n-1,$$

$$(ii) |\mu_n(E_n)| > \delta$$

and

$$(iii) \sum_{j=n+1}^\infty |\mu_n|(E_j) < \frac{\epsilon}{2^n}.$$

PROOF. Since  $K$  is not uniformly  $s$ -additive, there exists a disjoint sequence  $\{A_n\}$  in  $A$ , a sequence  $\{\lambda_n\}$  in  $K$ , and a  $\delta > 0$  such that

$$|\lambda_n(A_n)| > \delta, \quad \text{for all } n.$$

Let  $\epsilon > 0$  be given.

We define by induction a decreasing sequence

$N = N_0 > N_1 > N_2 > \dots$  of infinite sets, and an increasing sequence  $1 = k_0 < k_1 < k_2 < \dots$  of integers such that  
for every  $i$

$$(a) \quad k_{i+1} \in N_i$$

$$(b) \quad |\lambda_\ell|(A_{k_i}) < \frac{\epsilon}{2^i}, \quad \forall \ell \in S_i$$

and

$$(c) \quad \sum_{j=i}^{\infty} |\lambda_{k_j}|(A_j) < \frac{\epsilon}{2^i}, \quad \forall i \in N_i.$$

Start by letting  $N_0 = N$  and  $k_0 = 1$ . Now assume  $N_{i-1}$  and  $k_{i-1}$  have been defined.

We claim that there exists some  $k_i > k_{i-1}$  with  $k_i$  in  $N_{i-1}$  such that  $|\lambda_\ell|(A_{k_i}) < \frac{\epsilon}{2^i}$  for infinitely many  $\ell$ 's in  $N_{i-1}$ . For if not, then for every  $j$  in  $N_{i-1}$ ,  $j > k_{i-1}$ ,  $|\lambda_\ell|(A_j)$  is  $\geq \frac{\epsilon}{2^i}$  for  $\ell$  sufficiently large, and it follows that for any  $m \in \mathbb{N}$  there exists  $\ell$  large enough that  $|\lambda_\ell|(A_j) \geq \frac{\epsilon}{2^i}$  for  $m$  distinct  $A_j$ 's. Then  $|\lambda_\ell|(\Omega) \geq \frac{m\epsilon}{2^i}$ , and since  $m$  is arbitrary this would violate the boundedness of  $K$ . Thus there must exist such a  $k_i$  and we have

$$|\lambda_{k_1}|(A_{k_1}) < \frac{\epsilon}{2^1}, \quad \forall \ell \in J,$$

where  $J$  is an infinite subset of  $N_{i-1}$ .

Next, since  $|\lambda_{k_1}|$  is  $\sigma$ -additive, we must have

$$\sum_{j=\ell}^{\infty} |\lambda_{k_1}|(A_j) \rightarrow 0 \quad \text{as } \ell \rightarrow \infty,$$

and hence

$$\sum_{j=1}^{\infty} |\lambda_{k_1}|(A_j) < \frac{\epsilon}{2^1}, \quad \forall \ell \in I,$$

where  $I$  is a final subset of  $N_i$ . Let  $N_i = I \cap J$ .

To complete the proof, let  $\mu_n = \lambda_{k_n}$  and  $E_n = A_{k_n}$  for  $n = 1, 2, 3, \dots$ . Then

$$(i) \quad |\mu_n|(E_j) = |\lambda_{k_n}|(A_{k_j}) < \frac{\epsilon}{2^j} \quad \text{for } j = 1, \dots, n-1$$

since  $1 \leq j < n = k_n \in N_{n-1} \subset N_j$ ,

$$(ii) \quad |\mu_n(E_n)| = |\lambda_{k_n}(A_{k_n})| > \delta,$$

and

$$(iii) \quad \sum_{j=n+1}^{\infty} |\mu_n|(E_j) = \sum_{j=n+1}^{\infty} |\lambda_{k_n}|(A_{k_j})$$

$$\leq \sum_{j=k_{n+1}}^{\infty} |\lambda_{k_n}|(A_j) < \frac{\epsilon}{2^n}$$

since  $k_{n+1}$  is in  $N_n$ .

4.2.10 COROLLARY. If  $K$  is a bounded subset of  $\text{ba}(A)$  which is not uniformly  $s$ -additive, then there exists  $\delta > 0$  such that for every  $\epsilon > 0$  there is a sequence  $\{\mu_n\}$  in  $K$  and a disjoint sequence  $\{E_n\}$  in  $A$  such that for every  $n$

$$i) |\mu_n(E_n)| > \delta$$

and

$$(ii) \sum_{j \neq n} |\mu_n|(E_j) < \epsilon.$$

4.2.11 Recall that  $c_0$  is the Banach space of all sequences

$a = \{a_n\} \subset \mathbb{R}$  with  $\lim_{n \rightarrow \infty} a_n = 0$  and with  $\|a\|_\infty = \sup_n |a_n|$ .

The dual  $c_0^*$  of  $c_0$  is  $\ell^1$  in a natural way. If  $n$  is in  $\mathbb{N}$ , then  $e_n$  denotes the  $n^{\text{th}}$  unit basis vector; i.e.,  $e_n(m) = \delta_{mn}$ . We let  $c'_*$  denote the subspace of  $c_0^*$  consisting of all finitely non-zero sequences. Then  $c'_*$  is a dense subspace of  $c_0^*$ .

4.2.12 THEOREM. Let  $v: A \rightarrow X$  be a bounded finitely additive measure.

If  $v$  is not  $s$ -additive, then there exists an isomorphism on the space  $c_0$  into  $X$ ; in fact, we can choose a disjoint sequence  $\{E_n\}$  in  $A$  such that  $\phi(e_n) = v(E_n)$ .

PROOF. If  $v$  is not  $s$ -additive, then the set  $\{x^* \circ v : x^* \in X^*, \|x^*\| \leq 1\}$  is not uniformly  $s$ -additive. Since  $v$  is bounded, there exists  $M$  such that  $|x^* \circ v|(\Omega) \leq M$  for all  $\|x^*\| \leq 1$ . By 4.2.10, there exists  $\delta > 0$ , a disjoint sequence  $\{E_n\}$  in  $A$ , and a sequence  $\{x_n^*\}$  in the unit ball of  $X^*$  such that

$$|x_n^* \circ v(E_n)| > \delta$$

and

$$\sum_{j \neq n} |x_n^* \circ v|(E_j) < \frac{\delta}{2}.$$

If  $a = (a_n)_{n=1}^\infty$  is in  $c_\infty$ , let

$$\varphi_a(x) = \sum_{j=1}^\infty a_j v(E_j) = \sum_{a_j \neq 0} a_j v(E_j).$$

Clearly  $\varphi_a$  is linear. If  $\|x^*\| \leq 1$ , then

$$\begin{aligned} |x^* \circ \varphi_a(x)| &\leq \sum_{j=1}^\infty |a_j| |x^* \circ v(E_j)| \\ &\leq \|a\| \|x^* \circ v\|(\Omega) \leq M \|a\|_\infty, \end{aligned}$$

so that  $\varphi_a$  is continuous. Note also that for every  $n$ ,

$$\begin{aligned}
 |x_n^* \circ \varphi_\infty(a)| &= \left| \sum_{\substack{a_j \neq 0 \\ j \neq n}} a_j x_n^* \circ v(E_j) \right| \\
 &\leq \|a_n x_n^* \circ v(E_n)\| = \left| \sum_{\substack{a_j \neq 0 \\ j \neq n}} a_j x_n^* \circ v(E_j) \right| \\
 &\leq \|a_n\| \delta - \|a\|_\infty \sum_{j \neq n} |x_n^* \circ v|(E_j) \\
 &\leq \|a_n\| \delta - \|a\|_\infty \cdot \frac{\delta}{2}.
 \end{aligned}$$

Taking the supremum over all  $x_n^*$  gives

$$\|\varphi_\infty(a)\| \geq \frac{\delta}{2} \|a\|_\infty.$$

Therefore  $\varphi_\infty : c_\infty \rightarrow X$  is an isomorphism of  $c_\infty$  onto  $Y = \varphi_\infty(c_\infty)$ . Since  $c_\infty$  is dense in  $c_0$ , there exists a unique extension of  $\varphi_\infty$  to a continuous linear map  $\varphi : c_0 \rightarrow \bar{Y}$ , and similarly there is a unique extension of  $\varphi_\infty^{-1}$  to a continuous linear map  $\psi : \bar{Y} \rightarrow c_0$ . By density,  $\psi = \varphi^{-1}$ , and thus  $\varphi$  is a topological isomorphism.

**4.2.13 COROLLARY (Bessaga-Pelczynski).** For a Banach space  $X$ , the following three statements are equivalent.

(1) For every algebra  $A$  every bounded finitely additive measure  $\mu : A \rightarrow X$  is s-additive.

(2) Whenever  $\sum_{n=1}^{\infty} x_n$  is a series in  $X$  such that

$\sum_{n=1}^{\infty} |x^*(x_n)| \leq \infty$  for all  $x^*$  in  $X^*$ , the series

$\sum_{n=1}^{\infty} x_n$  is unconditionally convergent in norm.

(3)  $X$  does not contain an isomorphic copy of  $c_0$ .

PROOF. (1)  $\Rightarrow$  (2). Let  $A$  be the algebra of finite and cofinite subsets of  $\mathbb{N}$ . Given a series  $\sum_{n=1}^{\infty} x_n$  as in (2), define  $v$  on  $A$  by

$$v(\{n_1, \dots, n_p\}) = \sum_{i=1}^p x_{n_i}$$

$$\text{and } v(N \setminus \{n_1, \dots, n_p\}) = - \sum_{i=1}^p x_{n_i}.$$

Then  $v$  is finitely additive.

Now the map from  $X^*$  to  $\ell^1$  given by  $x^* \mapsto \{x^*(x_n)\}_{n=1}^{\infty}$  is well-defined, linear, and has a closed graph. Thus it is continuous, so that we have

$$\sup_{E \in A} \|v(E)\| = \sup_{\|x^*\| \leq 1} \sup_{\substack{E \in A \\ E \text{ finite}}} \left| \sum_{n \in E} x^*(x_n) \right|$$

$$\leq \sup_{\|x^*\| \leq 1} \sum_{n=1}^{\infty} |x^*(x_n)| < \infty.$$

Hence  $v$  is bounded. By (1),  $v$  is s-additive and it follows

that  $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} v(\{n\})$  converges unconditionally in norm.

(2)  $\Rightarrow$  (3) since the series  $\sum_{n=1}^{\infty} x_n$ , where  $x_n = \delta_n$ ,  
is not convergent in  $c_0$  but otherwise satisfies (2).

(3)  $\Rightarrow$  (1) by 4.2.12.

4.2.14 COROLLARY. Let  $v:A \rightarrow X$  be a bounded finitely additive measure. Then  $v$  is s-additive if and only if for every monotone increasing (resp., decreasing) sequence in  $A$ ,  
 $\lim_{n \rightarrow \infty} v(E_n)$  exists in the weak topology.

PROOF. ( $\Rightarrow$ ) by 4.2.5.

( $\Leftarrow$ ). If  $v$  is not s-additive, select a disjoint sequence  $\{A_n\} \subset A$  and an isomorphism  $\varphi:c_0 \rightarrow X$  such that  $\varphi(\delta_n) = v(A_n)$ .

Then

$$\varphi\left(\sum_{n=1}^m \delta_n\right) = \sum_{n=1}^m v(A_n) = v\left(\bigcup_{n=1}^m A_n\right).$$

Since  $\text{wk-lim}_{m \rightarrow \infty} \sum_{n=1}^m \delta_n$  does not exist in  $c_0$ , and since  $\varphi$  is an isomorphism,  $\text{wk-lim}_{m \rightarrow \infty} v\left(\bigcup_{n=1}^m A_n\right)$  does not exist.

#### 4.3 THE STONE COMPLETION OF AN ALGEBRA

##### 4.3.1 Let $A$ be any algebra of subsets of a set $\Omega$ .

Let  $\Omega_0$  be the set of all  $\{0,1\}$ -valued finitely additive measures  $\varepsilon$  on  $A$  such that  $\varepsilon(\Omega) = 1$ , and give  $\Omega_0$  the topology of elementwise convergence on  $A$ . Then  $\Omega_0$  is a closed subset of  $\{0,1\}^A$  and so  $\Omega_0$  is a totally disconnected compact Hausdorff space.

Define  $\varphi: A \rightarrow \Omega_0$  by  $\varphi(E) = \{\varepsilon \in \Omega_0 : \varepsilon(E) = 1\}$ . We show that  $\varphi$  is a Boolean algebra isomorphism from  $A$  onto the algebra of clopen subsets of  $\Omega_0$ .

(1)  $\lambda(E)$  is a clopen subset of  $\Omega_0$  for every  $E$  in  $A$ .

The map  $\hat{\varepsilon}:\Omega_0 \rightarrow \{0,1\} \subset \mathbb{R}$  given by  $\hat{\varepsilon}(\varepsilon) = \varepsilon(E)$  is continuous on  $\Omega_0$  by the choice of topology on  $\Omega_0$ . Hence

$$\varphi(E) = \hat{\varepsilon}^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) = \hat{\varepsilon}^{-1}\left(\left[\frac{1}{2}, \frac{3}{2}\right]\right)$$

is clopen in  $\Omega_0$ .

(2)  $\varphi$  is a one-to-one function.

If  $E \neq F$ , then there exists  $x$  in  $E \Delta F$ . Then  $\varepsilon_x(E) \neq \varepsilon_x(F)$ , where  $\varepsilon_x \in \Omega_0$  is point mass at  $x$ . Hence  $\varphi(E) \neq \varphi(F)$ .

(3)  $\varphi$  is an algebra isomorphism of  $A$  into the algebra  $A_0$  of clopen sets in  $\Omega_0$ .

Given  $e \in \Omega_0$ , let

$$A_e = \{E \in A : e(E) = 1\}.$$

Then  $A_e$  is a filter in  $A$  (closed under intersections and under supersetting by members of  $A$ ) such that if  $E$  is in  $A$ , then either  $E$  is in  $A_e$  or  $\Omega \setminus E$  is in  $A_e$ . We have  $e \in \varphi(E) \Leftrightarrow E \in A_e$ . If  $E, F$  are in  $A$ , then

$$e \in \varphi(E \cap F) \Leftrightarrow E \cap F \in A_e$$

$\Leftrightarrow E$  and  $F$  are in  $A_e$

$\Leftrightarrow e \in \varphi(E) \cap \varphi(F)$ ,

so  $\varphi(E \cap F) = \varphi(E) \cap \varphi(F)$ . Next,

$$e \in \varphi(\Omega \setminus E) \Leftrightarrow e(\Omega \setminus E) = 1$$

$\Leftrightarrow e(E) = 0$

$\Leftrightarrow e \in \Omega_0 \setminus \varphi(E)$ ,

so  $\varphi(\Omega \setminus E) = \Omega_0 \setminus \varphi(E)$ . It follows that  $\varphi(\Omega \setminus F) = \varphi(E) \setminus \varphi(F)$  and  $\varphi(E \cup F) = \varphi(E) \cup \varphi(F)$ .

(4)  $\varphi(A)$  is the set of all clopen subsets of  $\Omega_0$ .

By (3),  $\varphi(A)$  is an algebra of clopen sets. Let  $Y$  denote the collection of all  $\varphi(A)$ -simple functions  $f = \sum_{i=1}^n a_i X_{\varphi(E_i)}$ . Then  $Y$  is a subalgebra of  $C(\Omega_0)$ , it separates points of  $\Omega_0$ , and it contains the constant functions. By the Stone-Weierstrass theorem,  $Y$  is uniformly dense in  $C(\Omega_0)$ . If  $A \subset \Omega_0$  is clopen, then  $X_A$  is in  $C(\Omega_0)$  so there exists  $f = \sum_{i=1}^n a_i X_{\varphi(E_i)}$  in  $Y$  such that

$$\left| \sum_{i=1}^n a_i X_{\varphi(E_i)}(e) - X_A(e) \right| < \frac{1}{4}, \quad \forall e \in \Omega_0.$$

We may assume  $\{\varphi(E_1), \dots, \varphi(E_n)\}$  forms a partition of  $\Omega_0$  and  $a_i \neq a_j$  for  $i \neq j$ . If  $A \cap \varphi(E_1) \neq \emptyset$ , say  $e \in A \cap \varphi(E_1)$ , then

$$\frac{1}{4} > \left| \sum a_i X_{\varphi(E_i)}(e) - X_A(e) \right| = |a_1 - 1| = a_1 > \frac{3}{4}.$$

Now if  $e_1 \notin \varphi(E_1) \setminus A$ , then

$$\frac{1}{4} > \left| \sum a_i X_{\varphi(E_i)}(e_1) - X_A(e_1) \right| = |a_1 - 0| = a_1 < \frac{1}{4},$$

and it follows that  $A \cap \varphi(E_1) \neq \emptyset = \varphi(E_1) \subset A$ . Thus

$A = \bigcup \{\varphi(E_i) : \varphi(E_i) \cap A \neq \emptyset\}$  is in  $\varphi(A)$ .

Therefore,  $\varphi:A \rightarrow A_0$  is a Boolean algebra isomorphism of  $A$  onto the algebra of all clopen subsets of the totally disconnected compact Hausdorff space  $\Sigma_0$ . The pair  $(A_0, \varphi)$  will be called the Stone Completion of  $A$ .

Note that if  $\mu:A \rightarrow X$  is finitely additive, then  $\mu_0 = \mu \circ \varphi^{-1}:A_0 \rightarrow X$  is finitely additive on  $A_0$  and hence is countably additive on  $A_0$  by 2.4.11.

## 4.4 WEAK COMPACTNESS - CONTROL MEASURES

4.4.1 Note that if  $(G, \varphi)$  is the Stone completion of  $G$ , then the map  $\mu \rightarrow \tilde{\mu} = \mu \circ \varphi^{-1}$  is an isometric isomorphism of  $ba(G)$  onto  $ba(G_0)$ ,  $ba(G_0) = ca(G_0)$  by 2.4.11, and by 4.14,  $ca(G_0)$  is isometrically isomorphic to  $ca(\mathcal{B}(\Omega_0))$  under  $r^{-1}$ . Hence  $ba(G)$  is isometrically isomorphic to  $ca(\mathcal{B}(\Omega_0))$  under  $\mu \rightarrow r^{-1}(\mu \circ \varphi^{-1})$ ; this isomorphism also preserves the lattice operations.

Theorem 3.2.16 gives conditions for weak compactness in  $ca(\mathcal{B}(\Omega_0))$ . In order to carry these results over to  $ba(G)$ , we shall need the following two lemmas.

As in §3.3, we say that a member  $\lambda$  of  $ba(G)$  is a (uniform) control measure for a subset  $K$  of  $ba(G)$  provided the members of  $K$  are (uniformly) absolutely continuous with respect to  $|\lambda|$ .

4.4.2. LEMMA. If  $G_0$  is the algebra of clopen subsets of a totally disconnected  $CT_2$ -space  $\Omega_0$ , and if  $r: ca(\mathcal{B}(\Omega_0)) \rightarrow ca(G_0)$  is the restriction map, then a set  $K \subset ca(\mathcal{B}(\Omega_0))$  is uniformly countably additive on  $B(\Omega_0)$  if and only if  $r(K)$  is uniformly  $s$ -additive on  $G_0$ .

PROOF. ( $\Rightarrow$ ) trivially.

( $\Leftarrow$ ). Suppose  $r(K)$  is uniformly  $s$ -additive on  $\mathcal{G}_0$ . By 4.2.8, it is sufficient to consider only the case when the members of  $K$  are non-negative. To prove the lemma in this case, we first establish the following fact. If  $C$  is any compact  $G_0$  in  $\Omega$ , then for every  $\epsilon > 0$  there is a clopen set  $V \supset C$  such that  $\mu(V \setminus C) < \epsilon$  for all  $\mu$  in  $K$ . For let  $C = \bigcap_{n=1}^{\infty} U_n$  where  $U_n \downarrow$  and each  $U_n$  is an open set. Since  $C$  is compact and  $\Omega$  is totally disconnected, we may assume each  $U_n$  is clopen. Since  $r(K)$  is uniformly  $s$ -additive,  $\lim_{n \rightarrow \infty} \mu(U_n) = \mu(C)$  uniformly over  $\mu$  in  $K$ .

Now suppose  $K$  is not uniformly countably additive. Then there exist a disjoint sequence  $(E_n)$  in  $\mathcal{B}(\Omega)$ , a sequence  $(\mu_n)$  in  $K$ , and an  $\epsilon > 0$  such that

$$\mu_n(E_n) > \epsilon, \quad \forall n.$$

By regularity (2.4.6), for each  $n$  there is a compact  $G_0$  set  $C_n$  such that  $C_n \subset E_n$  and

$$\mu_n(C_n) > \epsilon, \quad \forall n.$$

Choose a clopen set  $V_1$  such that  $C_1 \subset V_1$  and  $\mu(V_1 \setminus C_1) < \frac{\epsilon}{4}$  for all  $\mu$  in  $K$ . Note that

$$V_2, C_2 \setminus V_1, C_3 \setminus V_1, \dots$$

are disjoint compact  $G_\delta$ 's,  $\mu_1(V_1) > \epsilon$ , and  $\mu_n(C_n \setminus V_1) > \frac{3\epsilon}{4}$  for all  $n \geq 2$ .

Next, choose  $V_2$  clopen such that

$$C_2 \setminus V_1 \subset V_2 \subset (G_0 \setminus V_1) \text{ and } \mu(V_2 \setminus (C_2 \setminus V_1)) < \frac{\epsilon}{8}$$

for all  $\mu$  in  $K$ . Note that

$$V_1, V_2, C_3 \setminus (V_1 \cup V_2), C_4 \setminus (V_1 \cup V_2), \dots$$

are disjoint compact  $G_\delta$ 's,  $\mu_1(V_1) > \epsilon$ ,

$\mu_2(V_2) > \frac{3\epsilon}{4}$ , and  $\mu_n(C_n \setminus (V_1 \cup V_2)) > \frac{5}{8}\epsilon$  for all  $n \geq 3$ . By induction there is a disjoint sequence

$V_1, V_2, V_3, \dots$  of clopen sets such that  $\mu_n(V_n) > \frac{\epsilon}{2}$

for all  $n$ . This contradicts the assumption that

$r(\lambda)$  is uniformly  $s$ -additive on  $G_0$ .

4.4.3. LEMMA. If  $G_0$  and  $G_0$  are as in 4.4.2, and if  $K$  is a subset of  $ca(\mathcal{B}(G_0))$ , then  $\lambda \in ca(\mathcal{B}(G_0))$  is a (uniform) control measure for  $K$  if and only if  $r(\lambda)$  is a (uniform) control measure for  $r(K)$ .

PROOF. ( $\Rightarrow$ ) trivially

( $\Rightarrow$ ) Let  $\mu$  be in  $K$  and suppose for every  $\epsilon > 0$  there exists  $\delta_\mu(\epsilon) > 0$  such that  $E \in G_0$ ,  $|\lambda|(E) < \delta_\mu(\epsilon) = |\mu|(E) \leq \epsilon$ .

We show that this same implication holds for  $E$  in  $\mathcal{B}(\Omega)$  and  $|\lambda|(E) < \delta_\mu(\epsilon)$ . Let  $C$  be any compact  $G_\delta$  with  $C \subset E$ . Then  $|\lambda|(C) < \delta_\mu(\epsilon)$  so by regularity there is a clopen set  $F \in G_0$  with  $C \subset F$  and  $|\lambda|(F) < \delta_\mu(\epsilon)$ ; hence  $|\mu|(F) \leq \epsilon$ , so  $|\mu|(C) \leq \epsilon$ . By regularity,  $|\mu|(E) = \sup_{C \text{ compact } G_\delta, C \subset E} (|\mu|(C)) \leq \epsilon$ .

4.4.4. THEOREM. If  $K$  is a subset of  $\text{ba}(G)$ , the following three statements are equivalent.

- (1)  $K$  is conditionally weakly compact in  $\text{ba}(G)$ .
- (2)  $K$  is bounded and uniformly absolutely continuous with respect to some  $\lambda$  in  $\text{ba}(G)$ .
- (3)  $K$  is bounded and uniformly s-additive on  $G$ .

PROOF. Note that if  $(G_0, \varphi)$  is the Stone completion of  $G$ , then each of (1), (2), and (3) is equivalent to the corresponding statement about  $K^1 = \{\mu = \varphi^{-1}: \mu \in K\}$

as a subset of  $ba(G_0)$ . By the lemmas and 4.1.4, each of (1), (2), and 3) is equivalent to the corresponding statement about  $r^{-1}(K^1)$  as a subset of  $ca(\mathcal{B}(\Omega))$ . An application of 3.2.16 completes the proof.

4.4.5. COROLLARY. If  $\Sigma = \Sigma(0)$ , and if  $K$  is a bounded subset of  $ca(\Sigma)$ , then  $K$  is uniformly countable additive on  $\Sigma$  if and only if  $r(K) = \{\mu|G : \mu \in K\}$  is uniformly s-additive on  $G$ .

4.4.6. THEOREM. If  $K$  is any subset of  $ba(G)$ , then the following four statements are equivalent.

- (1)  $K$  has a control measure in  $ba(\Sigma)$ .
- (2) There is a sequence  $(\mu_n)$  in  $K$  such that

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu_n}{1 + \|\mu_n\|}$$

is a control measure for  $K$ .

- (3) There is a control measure for  $K$  in  $\overline{co}(K)$ .
- (4) The set of control measures for  $K$  in  $\overline{co}(K)$  is dense in  $\overline{co}(K)$ .

PROOF. As in the proof of 4.4.4., using 3.3.3.

4.4.7. COROLLARY. If  $K$  is a conditionally weakly compact subset of  $\text{ba}(G)$ , then there exists a uniform control measure in  $\overline{\text{co}}(K)$ . In fact, the set of uniform control measures in  $\overline{\text{co}}(K)$  is dense in  $\overline{\text{co}}(K)$ . Every control measure for  $K$  is a uniform control measure for  $K$ .

4.4.8. EXAMPLE. Let  $G$  be the algebra of finite and cofinite subsets of  $N$ , and let  $\mu_n$  denote point mass at  $n$ . Then for every  $E$  in  $G$ ,

$$\mu(E) = \lim_{n \rightarrow \infty} \mu_n(E) = \begin{cases} 0 & \text{if } E \text{ is finite,} \\ 1 & \text{if } E \text{ is cofinite,} \end{cases}$$

exists. Note that  $(\mu_n)$  is not uniformly  $s$ -additive on  $G$  (and hence  $\mu_n$  does not converge to  $\mu$  weakly), and  $\mu$  is not countably additive. Hence obvious conjectures for generalizations of 3.2.8, 3.2.9, and 3.2.11 fail by this example. However, see 5.1.8 ff. below.

#### R.4.1. REMARKS AND REFERENCES

The results here are classical; see e.g. Dunford and Schwartz [1958].

#### R.4.2. REMARKS AND REFERENCES

- (1) It should be emphasized that for finitely additive measures,  $\mu \ll \nu$  is not equivalent to  $|\nu|(\mathbb{E}) = 0 = \mu(\mathbb{E}) = 0$ .
- (2) The important notion of  $s$ -additivity was introduced by Rickart [1943] who used the term "strong boundedness" (or " $s$ -boundedness"). This terminology is prevalent in the literature, but we prefer " $s$ -additivity". Another term often used is "exhausting". Corollary 4.2.7 is due to Rickart [1943].
- (3) Theorem 4.2.9 (4.2.10) is a modification by Uhl [1973] of a lemma of Rosenthal [1968, Lemma 1]. (Rosenthal's lemma is essentially 5.1.14 below.) This theorem seems to be the basis of the so-called "sliding hump method" of proof.
- (4) Theorem 4.2.12 (4.2.13) is immediate from results of Bessaga and Pelczynski [1958], but their application to  $s$ -additivity of vector measures was first observed by Diestel [1973(a)]. The proof here is from Uhl [1973]. In connection with this result, see also 5.1.15 and 5.1.16 below.
- (5) Corollary 4.2.14 follows quickly from results of Kluyanek [1973]. The proof here is from Uhl [1973]. It is noted in §5.2 that 4.2.14 is a generalization of a basic theorem of Pettis and Orlicz (5.2.4 below).

Additional References: Brooks [1973], Hoffmann-Jorgensen [1971].

R.4.3. REMARKS AND REFERENCES.

The result here is so basic that it is now part of the folklore of analysis (see Dunford and Schwartz [1958, p. 312]); it goes back to Stone [1937].

R.4.4. REMARKS AND REFERENCES.

- (1) The idea of using the Stone completion to carry results about  $ca(\Sigma)$  to results about  $ba(A)$  occurs in Dunford and Schwartz [1958, p. 314] where the equivalence of (1) and (2) in 4.4.4 is obtained.
- (2) Lemma 4.4.2 (and 4.4.5) seems to have been first pointed out in the literature by Brooks [1971]; the proof given here is more direct and elementary than that indicated by Brooks.
- (3) Theorem 4.4.4 is intimately connected with 5.1.5 below. See R.5.1 for references for 5.1.5.
- (4) Theorems 4.4.6 and 4.4.7 are due to Huff and Morris [1973].