

3. THE SPACE $ca(\Sigma)$.

3.1 VARIATION = ABSOLUTE CONTINUITY

3.1.1 If $\mu: \Sigma \rightarrow \mathbb{R}$ is c.a., a set $E \in \Sigma$ is said to be a positive domain (respectively, negative domain, null set) with respect to μ provided $E \supset F \in \Sigma$ implies $\mu(F) > 0$ (resp., $\mu(F) \leq 0$, $\mu(F) = 0$). The positive domains (negative domains, null sets) form a σ -ideal in Σ .

3.1.2 PROPOSITION. If $\mu: \Sigma \rightarrow \mathbb{R}$ is c.a., if E is in Σ , and if $\mu(E) > 0$, then there exists a positive domain $F \subset E$ such that $\mu(F) > 0$.

PROOF. Suppose not. Let n_1 be the smallest positive integer such that there exists $E_1 \subset E$ with $\mu(E_1) < -\frac{1}{n_1}$.

By induction, let n_k be the smallest positive integer such that there exists $E_k \subset E \setminus (\bigcup_{l=1}^{k-1} E_l)$ with

$\mu(E_k) < -\frac{1}{n_k}$. Let $F = E \setminus (\bigcup_{l=1}^{\infty} E_l)$. Then $\mu(F) > \mu(E) > 0$.

Since $\sum \mu(E_k) = \mu(\bigcup E_k)$ is in \mathbb{R} , $\mu(E_k) \sim 0$ and so $n_k \rightarrow \infty$. Now suppose $C \subset F$ and $\mu(C) < 0$. Choose

k such that $\mu(G) < -\frac{1}{n_k - 1}$. This contradicts the

choice of n_k since $G \subseteq E \setminus (\bigcup_{i=1}^{k-1} E_i)$ and $\mu(G) < -\frac{1}{n_k - 1}$.

3.1.3. COROLLARY. (The Hahn Decomposition Theorem). If $\mu: \Sigma \rightarrow \mathbb{R}$ is c.s., then there exist sets $A, B \in \Sigma$ such that (i) A is a positive domain, (ii) B is a negative domain, (iii) $A \cap B = \emptyset$, and (iv) $A \cup B = \Omega$. Moreover, A and B are unique up to null sets (i.e., if A' and B' is another such pair, then $A \Delta A'$ and $B \Delta B'$ are null sets.)

PROOF. Let $s = \sup \{\mu(F): F \text{ is a positive domain}\}$, and choose F_n 's positive domains such that $\mu(F_n) \nearrow s$.

Let $A = \bigcup_{n=1}^{\infty} F_n$. Then A is a positive domain, and so $\mu(A) = s$. Let $B = \Omega \setminus A$. Then B must be a negative domain by 3.1.2. If A' , B' are another such pair; then $A \Delta A'$ and $B \Delta B'$ are both positive and negative domains, hence null sets.

3.1.4. COROLLARY. If $\mu: \Sigma \rightarrow \mathbb{R}$ is c.s., then μ is bounded; i.e., μ is in $\text{ca}(\Sigma)$.

3.1.5. COROLLARY. If $\mu: \Sigma \rightarrow X$ is c.s., where X is any Banach space, then μ is bounded.

PROOF. By the uniform boundedness principle.

3.1.6. If μ is in $ca(\Sigma)$ and A, B are as in 3.1.3, let

$$\mu^+(E) = \mu(E \cap A)$$

$$\mu^-(E) = \mu(E \cap B) \quad (E \in \Sigma).$$

$$|\mu|(E) = \mu^+(E) + \mu^-(E)$$

Then μ^+ , μ^- , and $|\mu|$ are in $ca^+(\Sigma)$ and $\mu = \mu^+ - \mu^-$.

The decomposition $\mu = \mu^+ - \mu^-$ is called the Jordan decomposition of μ . The measure $|\mu|$ is called the (total) variation of μ . The (total) variation norm of μ is defined to be the real number $\|\mu\| = |\mu|(\Omega)$.

3.1.7. PROPOSITION. (1) $\mu^+(E) = \sup \{\mu(F) : F \in \Sigma, F \subset E\}$

$$(11) \quad \mu^-(E) = -\inf \{\mu(F) : F \in \Sigma, F \subset E\}$$

$$(111) \quad |\mu|(E) \leq \mu^+(E) + \mu^-(E) = |\mu|(E) \leq 2 \sup \{|\mu(F)| : F \in \Sigma, F \subset E\}.$$

PROOF. Trivial.

3.1.8. LEMMA. If σ and λ are in $ca^+(\Sigma)$, if $\sigma \neq 0$, and

if $\lambda(E) = 0 \Rightarrow \sigma(E) = 0$, then there exist $a > 0$ and $A \in \Sigma$ such that

$$(1) \quad \lambda(A) > 0$$

and

$$(11) \quad a \lambda(E \cap A) \leq \sigma(E), \forall E \in \Sigma.$$

PROOF. Since $\sigma(\Omega) \neq 0$, there exists $k > 0$ such that $k\sigma(\Omega) = \lambda(\Omega) > 0$.

By 3.1.2, there exists $A \in \Sigma$ such that $(k\sigma - \lambda)(A) \geq 0$ (and thus $\lambda(A) \geq 0$) and

$$(k\sigma - \lambda)(A \cap E) \geq 0$$

$$k\sigma(A \cap E) \geq \lambda(A \cap E)$$

$$k\sigma(E) \geq \lambda(A \cap E)$$

$$\sigma(E) \geq \frac{1}{k} \lambda(A \cap E)$$

for all E in Σ .

3.1.9. THEOREM (The Radon-Nikodym Theorem). If λ is in $\text{ca}^+(\Sigma)$, if $\mu \in \text{ca}(\Sigma)$, and if $\lambda(E) = 0 \Rightarrow \mu(E) = 0$, then there exists an (essentially unique) f in $L^1(\lambda)$ such that

$$(*) \quad \mu(E) = \int_E f d\lambda, \quad \forall E \in \Sigma$$

Conversely, if f is in $L^1(\lambda)$ and μ is defined by

(*), then μ is in $\text{ca}(\Sigma)$, $\lambda(E) = 0 \Rightarrow \mu(E) = 0$,

$$\mu^+(E) = \int_E f^+ d\mu, \quad \mu^-(E) = \int_E f^- d\mu, \quad \text{and} \quad |\mu|(E) = \int_E |f| d\mu.$$

In particular, $\|\mu\| = \|f\|_1$.

PROOF. The converse part of the theorem follows from definitions.

If $\int_E g d\lambda = \int_E f d\lambda$, $\forall E \in \Sigma$ (some $f, g \in L^1(\lambda)$),
then $\int_E (f - g) d\lambda = 0$, $\forall E \in \Sigma$, and hence from above
 $\|f - g\|_1 = 0$, so $f = g$ a.e. Thus we have uniqueness.

Now given μ such that $\lambda(E) = 0 \Rightarrow \mu(E) = 0$,

to show the existence of f in $L^1(\lambda)$ satisfying

(*) it is sufficient by the Hahn-Jordan decomposition to assume $\mu \geq 0$, and we do assume that. Let P be the set of all non-negative $h \in L^1(\lambda)$ such that

$$\int_E h d\lambda \leq \mu(E), \quad \forall E \in \Sigma.$$

The set P is partially ordered by $h \leq g$ a.e. Let Q be a chain in P , and let $\alpha = \sup_{h \in Q} \|h\|_1$.

Then $0 \leq \alpha \leq \|h\|_1 < \infty$. Choose h_n in Q with $\|h_n\|_1/\alpha$. Since either $h_n \leq h_{n+1}$ or $h_n \geq h_{n+1}$ a.e., we have $h_n \leq h_{n+1}$ a.e., $\forall n$. By the monotone convergence theorem, $h_n / \alpha \in L^1(\lambda)$, a.e., and

$$\|h\|_1 = \int_E h d\lambda = \lim \int_E h_n d\lambda = \alpha$$

Also, h is in P . If g is in Q then either $g \leq h$ or $g \geq h$ a.e. But since $\int_E h d\lambda = \alpha \geq \int_E g d\lambda$, we must

have $s \leq h$. Thus h is an upper bound for σ .

Therefore, by Zorn's lemma there exists a maximal element f in P . We have

$$\mu_f(E) = \int_E f d\lambda \leq \mu(E),$$

and we wish to show equality. Note that if $\sigma = \mu = \mu_p$, then $\sigma \geq 0$ and $\lambda(E) = 0 \Rightarrow \sigma(E) = 0$. If $\sigma \neq 0$, then by the lemma there exist $A \in \Sigma$ and $c > 0$ such that $\lambda(A) \neq 0$ and

$$c\lambda(E \cap A) \leq \sigma(E) = \mu(E) = \mu_p(E), \quad \forall E \in \Sigma.$$

But this implies that for all E in Σ ,

$$\begin{aligned} \int_E (f + cX_A) d\lambda &= \mu_p(E) + c\lambda(E \cap A) \\ &\leq \mu_p(E) + \mu(E) = \mu_p(E) = \mu(E), \end{aligned}$$

which contradicts maximality of f .

3.1.10 COROLLARY. The variation norm $\|\cdot\|$ is a linear norm on $ca(\Sigma)$. It is equivalent to $\|\cdot\|_\infty$.

PROOF. If μ, ν are in $ca(\Sigma)$, then letting $\lambda = |\mu| + |\nu|$ we have by 3.1.9,

$$\|\mu + \nu\| = \|f + g\|_1 \leq \|f\|_1 + \|g\|_1 = \|\mu\| + \|\nu\|$$

for some $f, g \in L^1(\lambda)$. Also $\|\mu\| = 0 \Leftrightarrow \mu = 0$,

and $\|c\mu\| = |c| \|\mu\|$ for any $c \in \mathbb{R}$.

That $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$ follows from
3.1.7 (ii).

3.1.11. COROLLARY. If $ca(\Sigma, \lambda)$ denotes the subspace of $ca(\Sigma)$ consisting of all μ for which $\lambda(E) = 0 \Rightarrow \mu(E) = 0$, then $(ca(\Sigma, \lambda), \|\cdot\|)$ is a Banach space which is isometrically isomorphic to $L^1(\lambda)$, where $\mu \in ca(\Sigma, \lambda)$ corresponds to $f \in L^1(\lambda)$ provided

$$\mu(E) = \int_E f d\lambda, \quad \forall E \in \Sigma.$$

3.1.12. If μ is in $ca(\Sigma, \lambda)$, the number f of $L^1(\lambda)$ for which $\mu(E) = \int_E f d\lambda$, $\forall E \in \Sigma$, is called the Radon-Nikodym derivative of μ with respect to λ , and the notation $\frac{d\mu}{d\lambda} = f$ is often used.

3.1.13. If μ and λ are in $ca(\Sigma)$, we say that μ is absolutely continuous with respect to λ provided

$$\lim_{|\lambda|(E) \rightarrow 0} \mu(E) = 0; \text{ i.e., } \forall \epsilon > 0 \exists \delta > 0$$

such that $E \in \Sigma$, $|\lambda|(E) < \delta \Rightarrow |\mu(E)| < \epsilon$. We write $\mu \ll \lambda$ to denote this.

3.1.14. COROLLARY. If μ is in $ca(\Sigma)$ and λ is in $ca^+(\Sigma)$, then $\mu \ll \lambda$ if and only if $\lambda(\Sigma) = 0$
 $\Rightarrow \mu(\Sigma) = 0$.

PROOF. By 3.1.9 and 2.3.9 (2.3.10).

3.1.15. EXERCISES. (1) Show that $ca(\Sigma)$ is a vector lattice if μ^+ is as defined in 3.16. (see).

(2) Show that if Ω is any set, then $m(\Omega)^*$ is a vector lattice if $\varphi \geq \psi$ means that $\varphi - \psi$ is a positive linear functional (Hint: in , show that $\tilde{\varphi} = \varphi^*$.)

(3) In exercise (2) replace $m(\Omega)$ by $c(\Omega)$, where Ω is a compact Hausdorff space.

(4) THEOREM (The Riesz Representation Theorem). If Ω is a compact Hausdorff space, then $c(\Omega)^*$ is isometrically isomorphic to $(ca(B(\Omega)), \|\cdot\|)$, where $\varphi \in c(\Omega)^*$ and $\mu \in ca(B(\Omega))$ are related under the isometric isomorphism provided

$$\varphi(f) = \int f d\mu, \quad \forall f \in C(\Omega).$$

(Here $\int f d\mu$ is defined to be $(\int f d\mu^+ - \int f d\mu^-)$.)

(5) If μ, ν are in $ca(\Sigma)$, write $\mu \perp \nu$ provided $[\mu]$ and $[\nu]$ are supported on disjoint sets; i.e.,

there exist $A, B \in \Sigma$ with $A \cap B = \emptyset$, $A \cup B = \Omega$, $|\mu|(A) = 0$, and $|\nu|(B) = 0$. Prove the following.

THEOREM. (Lebesgue Decomposition Theorem). If μ , ν are in $c\alpha(\Sigma)$, then there exist unique μ_0, μ_1 in $c\alpha(\Sigma)$ such that

$$(1) \quad \mu = \mu_0 + \mu_1,$$

$$(11) \quad \mu_0 \perp v_0$$

end.

(111) $\mu_1 \ll v$

(Hint: Let $\lambda = |\mu| + |\nu|$ and take $\epsilon \geq 0$ such that
 $|\nu|(E) = \int_E g \lambda, \forall E \in \Sigma$. Consider the two sets
 $A = \{\omega : g(\omega) > 0\}$ and $B = \{\omega : g(\omega) = 0\}.$)

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3.2

CONVERGENCE OF SEQUENCES OF MEASURES.

3.2.1.

There are three notions of convergence of sequences of measures $\{\mu_n\} \subset ca(\Sigma)$ that are of importance to us:

(1) elementwise convergence; i.e.,

$\mu_n(E) \rightarrow \mu(E)$ for every E in Σ .

(2) weak convergence; i.e., $\mu_n \rightarrow \mu$

weakly in $ca(\Sigma)$.

(3) norm convergence; i.e., $\mu_n \rightarrow \mu$

in the $\|\cdot\|_1$ - (or the $\|\cdot\|_\infty$) norm.

Clearly (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow μ is finitely additive.

The study of these involves certain uniform behavior characteristics of the members of the sequence, and we first consider them.

Since some of the following proofs are unchanged for the case when the range of the measures lie in a Banach space, we state and prove them in this setting.

Throughout, X is a Banach space.

3.2.2.

Given λ a non-negative member of $ca(\Sigma)$, recall that there is a naturally defined pseudometric d on

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Σ given by $d(A, B) = \lambda(A \Delta B)$ (see §2.2). With it, Σ is a complete pseudometric space in which the set operations are all continuous. We say that a finitely additive function $\mu : \Sigma \rightarrow X$ is λ -continuous at $E \in \Sigma$ if it is d -continuous at E . A family K of f.a. functions $\mu : \Sigma \rightarrow X$ is

(i) equi- λ -continuous at $E \in \Sigma$ if and only if

$$\forall \epsilon > 0 \exists \delta > 0 \ni$$

$$E, F \in \Sigma, d(E, F) < \delta \Rightarrow ||\mu(E) - \mu(F)|| < \epsilon, \forall \mu \in K.$$

(ii) equi-uniformly- λ -continuous on Σ if and only if

$$\forall \epsilon > 0 \exists \delta > 0 \ni$$

$$E, F \in \Sigma, d(E, F) < \delta \Rightarrow ||\mu(E) - \mu(F)|| < \epsilon,$$

$$\forall \mu \in K.$$

Moreover, we say that the family K is uniformly countably additive provided for every decreasing sequence $\{E_n\}$ in Σ with $\cap E_n = \emptyset$, $\forall \epsilon > 0 \exists N \ni n \geq N \Rightarrow ||\mu(E_n)|| < \epsilon, \forall \mu \in K$.

3.2.3. PROPOSITION. Let λ be a non-negative member of $ca(\Sigma)$, and let K be a family of finitely additive functions $\mu : \Sigma \rightarrow X$. Then the following three statements are equivalent.

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(1) K is equi- λ -continuous at some E in Σ .

(2) K is equi- λ -continuous at \emptyset .

(3) K is equi-uniformly- λ -continuous on Σ .

Moreover, (1) - (3) imply

(4) K is uniformly countably additive on Σ .

PROOF. (3) \Rightarrow (2) \Rightarrow (1) clearly. Suppose (1) holds.

Given $\epsilon > 0$ choose $\delta > 0$ such that

$$B \in \Sigma, \lambda(B \Delta E) < \delta \Rightarrow ||\mu(B) - \mu(E)|| < \frac{\epsilon}{4},$$

$$\forall \mu \in K.$$

If $A \in \Sigma$ and $\lambda(A) < \delta$, then $\lambda((E \cup A) \Delta E) \leq \lambda(A) < \delta$

and $\lambda((E \setminus A) \Delta E) \leq \lambda(A) < \delta$, so

$$\begin{aligned} A \in \Sigma, \lambda(A) < \delta &\Rightarrow ||\mu(A)|| = ||\mu(A \cup E) - \mu(E \setminus A)|| \\ &= ||\mu(A \cup E) - \mu(E)|| + ||\mu(E) - \mu(E \setminus A)|| \\ &< \frac{\epsilon}{2}, \quad \forall \mu \in K. \end{aligned}$$

Next, if C, D are in Σ and

$\delta > \lambda(C \Delta D) = \lambda(C \setminus D) + \lambda(D \setminus C)$, then for all

μ in K

$$\begin{aligned} ||\mu(C) - \mu(D)|| &= ||\mu(C \setminus D) - \mu(D \setminus C)|| \\ &\leq ||\mu(C \setminus D)|| + ||\mu(D \setminus C)|| \\ &< \epsilon. \end{aligned}$$

Hence (1) \Rightarrow (3).

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Finally, if (2) holds and if $E_n \downarrow \phi$, then

$\forall \epsilon > 0 \exists \delta > 0$ and $\exists N$ such that

$$n \geq N \Rightarrow \lambda(E_n) < \delta \Rightarrow ||\mu(E_n)|| < \epsilon.$$

Hence K is uniformly countably additive.

3.2.4. COROLLARY. If λ is in $ca^+(\Sigma)$ and if $\mu : \Sigma \rightarrow X$ is finitely additive, then μ is λ -continuous at some $E \Leftrightarrow \mu$ is λ -continuous at $\phi \Leftrightarrow \mu$ is uniformly λ -continuous $\Leftrightarrow \mu \ll \lambda \Rightarrow \mu$ is countably additive.

3.2.5. If (1) - (3) of 3.2.3 hold, we simply say that the members of K are uniformly absolutely continuous with respect to λ and denote this by writing

$$\lim_{\lambda(E) \rightarrow 0} \mu(E) = 0 \text{ uniformly over } \mu \text{ in } K,$$

or simply

$$\lim_{\lambda(E) \rightarrow 0} K(E) = 0.$$

3.2.6. PROPOSITION. If K is any subset of $ca(\Sigma, X)$, the following six statements are equivalent.

(1) If $\{E_n\}$ is a disjoint sequence in Σ , then for every $\epsilon > 0$ there exists $N' \in \mathbb{N}$ such that

$$n \geq n \geq N \Rightarrow \left| \left| \sum_{i=n}^m \mu(E_i) \right| \right| < \epsilon, \forall \mu \in K.$$

(2) If $\{E_\mu\}$ is a disjoint sequence in Σ , then

for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \left| \left| \sum_{i=n}^{\infty} \mu(E_i) \right| \right| < \epsilon, \forall \mu \in K.$$

(3) If $\{E_\mu\}$ is a disjoint sequence in Σ , then for

every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \left| \left| \mu(E_n) \right| \right| < \epsilon, \forall \mu \in K.$$

(4) If $\{E_\mu\}$ is a monotone increasing sequence in

Σ , then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$

such that

$$\text{u}, n \geq N \Rightarrow \left| \left| \mu(E_n) - \mu(E_m) \right| \right| < \epsilon, \forall \mu \in K.$$

(5) If $\{E_\mu\}$ is a monotone decreasing sequence

in Σ , then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$

such that

$$\text{u}, n \geq N \Rightarrow \left| \left| \mu(E_n) - \mu(E_m) \right| \right| < \epsilon, \forall \mu \in K.$$

(6) K is uniformly countably additive on Σ .

PROOF. (1) \Leftrightarrow (2), and (1) \Rightarrow (3) trivially.

(3) \Rightarrow (1). Suppose $\{E_n\}$ is a disjoint sequence and that (1) does not hold. Then there exists $\epsilon > 0$ such that for all N there exist $n_N \geq m_N \geq N$ with

$$\left| \left| \mu_N \left(\bigcup_{i=m_N}^{n_N} E_i \right) \right| \right| = \left| \left| \sum_{i=m_N}^{n_N} \mu_N(E_i) \right| \right| \geq \epsilon$$

for some μ_N in K . Take $N = 1$, choose n_1, m_1

as above, and let $F_1 = \bigcup_{i=m_1}^{n_1} E_i$. Next, choose

$n_{m_1+1} > n_1 + 1$ as above, and let

$F_2 = \bigcup_{i=n_1+1}^{n_{m_1+1}} E_i$. By induction we get a disjoint

sequence F_1, F_2, \dots such that for every n ,

$\left| \left| \mu_n(F_n) \right| \right| \geq \epsilon$, for some μ_n in K . Thus (3) cannot hold.

(1) \Rightarrow (4). Suppose $E \uparrow$. Let

$E_n = E_{n+1} \setminus E_n$. Then the E_n 's are disjoint, so

by (1) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such

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that $m \geq n \geq N$ implies

$$\epsilon > \left| \left| \sum_{i=n}^m \mu(E_i) \right| \right| = \left| \left| \mu(E_n) - \mu(E_m) \right| \right|.$$

(4) \Rightarrow (5) by complementation;

(5) \Rightarrow (6) clearly, and (6) \Rightarrow (2) by definition.

3.2.7. THEOREM. Let (X, d) , (Y, p) be pseudo-metric spaces with (X, d) complete, let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous functions on X to Y , and suppose $f_n(x) \rightarrow f(x)$ for every x in X . Then

$$A = \{x \in X : \{f_n\} \text{ is equi-continuous at } x\}$$

is a residual set in X (i.e., A is the complement of a set of first category.)

PROOF. For every $\epsilon > 0$, let U_ϵ denote the set of all points x in X such that there exists $\delta > 0$ (depending on x) such that

$$y, z \in B_\delta(x) \Rightarrow p(f_n(x), f_n(y)) < \epsilon, \forall n,$$

where $B_\delta(x)$ is the open ball centered at x of radius δ . Clearly U_ϵ is open and $U_\epsilon \downarrow$ as $\epsilon \downarrow$.

We will show that U_ϵ is dense for $\epsilon > 0$. Thus

$$E = \bigcap_{\epsilon>0} U_\epsilon = \bigcap_{\epsilon>0} \bigcup_{k=1}^\infty U_{\frac{1}{k}} \text{ is a residual set in } X.$$

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is a residual set. If x_0 is in \mathbb{X} , then for

every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x_0, y) < \delta \Rightarrow p(f_n(x), f_m(y)) < \epsilon, \forall n;$$

i.e., $\{f_n\}$ is equi-continuous at x_0 . Hence

$\mathbb{B} \subset A$, so A is residual.

Thus, it is sufficient to show U_ϵ is dense

for arbitrary $\epsilon > 0$. Let B be any open ball
in \mathbb{X} . We are to show $U_\epsilon \cap B \neq \emptyset$. For

$m, n = 1, 2, \dots$, let

$$E_{m,n} = \{x \in \overline{B} : p(f_n(x), f_m(x)) \leq \frac{\epsilon}{2}\},$$

and let

$$E_p = \bigcap_{m, n \geq p} E_{m,n}.$$

Then E_p is closed, and by hypotheses $\overline{B} = \bigcup_{p=1}^{\infty} E_p$.

By the Baire category theorem, $\exists p$ such that E_p
contains a non-void relatively open ball $B_1 \cap \overline{B}$,

where B_1 is an open ball in \mathbb{X}). But

$B_1 \cap \overline{B} \neq \emptyset \Rightarrow B_1 \cap B \neq \emptyset$. Thus E_p contains some

open ball $B_2(x_0) \subset B_1$. Hence

$$d(x_0, y) < \delta \Rightarrow p(f_n(y), f_m(y)) \leq \frac{\epsilon}{2}, \forall m, n \geq p.$$

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Since f_p is continuous at x_0 , by reducing δ

if necessary, we may assume $\delta > 0$ and

$$d(x_0, y) < \delta \Rightarrow p(f_n(x_0), f_p(y))$$

$$\leq p(f_n(x_0), f_p(x_0)) + p(f_p(x_0), f_p(y))$$

$$+ p(f_p(y), f_n(y))$$

$$< \frac{\epsilon}{2} , \quad \forall n \geq p .$$

Again, by reducing δ if necessary, we may assume

$\delta > 0$ and

$$d(x_0, y) < \delta \Rightarrow p(f_n(x_0), f_n(y)) < \frac{\epsilon}{2} , \quad \forall n .$$

Then

$$y, z \in B_\delta(x_0) \Rightarrow p(f_n(y), f_n(z)) < \epsilon , \quad \forall n ,$$

and so x_0 is in $\bigcap U_\epsilon$.

3.2.8. COROLLARY. (The Vitali-Hahn-Saks Theorem). Let

λ be a non-negative member of $ca(\mathbb{I})$, let X

be a Banach space, and for each n , let

$\mu_n : \mathbb{I} \rightarrow X$ be countably additive and λ -continuous.

If for every E in \mathbb{I} , $\{\mu_n(E)\}_{n=1}^\infty$ converges to

some $\mu(E)$, then $\{\mu_n\}$ is uniformly absolutely

continuous with respect to λ . (In particular,

$\{\mu_n\}$ is uniformly countably additive and so μ is
countably additive.)

3.2.9 COROLLARY. (Nikodym's Theorem). Let (μ_n) be a sequence in $\text{ca}(\Sigma)$ such that $(\mu_n(E))_{n=1}^{\infty}$ converges to $\mu(E) \in \mathbb{R}$ for every E in Σ . Then (μ_n) is uniformly countably additive and μ is in $\text{ca}(\Sigma)$.

PROOF. Let $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_n|}{|\mu_n| + 1}$. Then λ is in $\text{ca}^+(\Sigma)$ and $\mu_n < \lambda$ for every n .

3.2.10. THEOREM. (Nikodym). Let K be a subset of $\text{ca}(\Sigma, X)$ such that for every E in Σ

$$\sup_{v \in K} \|v(E)\| < \infty.$$

Then K is uniformly bounded; i.e., $\sup_{E \in \Sigma} \sup_{v \in K} \|v(E)\| < \infty$.

PROOF. It is sufficient to prove the theorem when $X = \mathbb{R}$, for in the general case we can consider the set $K' = \{x^* \circ v : v \in K, x^* \in X^*, \|x^*\| \leq 1\}$. Thus we assume $X = \mathbb{R}$.

It is sufficient to prove that an arbitrary sequence $(\mu_n)_{n=1}^{\infty}$ in K is uniformly bounded. Define $\lambda \in \text{ca}^+(\Sigma)$ by

$$\lambda(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_n|(E)}{|\mu_n| + 1}, \quad (E \in \Sigma),$$

and consider Σ as a complete pseudo-metric space

with pseudo-metric $d(E, F) = \lambda(E \Delta F)$ (52.2).

Let

$$H_n = \{E \in \Sigma : |\mu_n(E)| \leq n \text{ for all } n = 1, 2, \dots\}.$$

Then H_n is d -closed in Σ and $\Sigma = \bigcup_{n=1}^{\infty} H_n$. By the

Baire category theorem there exists a set $E_0 \in \Sigma$

and an $\epsilon > 0$ such that

$$\lambda(E \Delta E_0) \leq \epsilon \Rightarrow |\mu_n(E)| \leq n_0, \quad \forall n,$$

for some fixed integer n_0 .

By 2.2.7 and 2.2.8 there exists a Σ -partition

$\{E_1, \dots, E_p\}$ of Ω such that for each i either

$\lambda(E_i) \leq \epsilon$ or E_i is an atom of λ . For convenience,

assume E_1, \dots, E_p are atoms of λ with

$$\lambda(E_i) \geq \epsilon \quad (i = 1, \dots, p).$$

Note that since E_i ($i = 1, \dots, p$) is an atom of λ ,

it is also an atom of $|\mu_n|$ for any n . Consequently,

if $F \subset E_i$ then either

$$|\mu_n|(E_i \setminus F) = 0 \text{ or } |\mu_n|(F) = 0, \text{ so either}$$

$$\mu_n(E_i) = \mu_n(F) \text{ or } \mu_n(F) = 0.$$

In particular, $E \subset E_i \Rightarrow |\mu_n(E)| \leq |\mu_n(E_i)|$ for

any $i = 1, \dots, p$ and any n .

Now let E be any fixed member of Σ , and let

$E_i = E \cap E_i$ ($i = 1, \dots, m$). By the previous

paragraph, if $1 \leq i \leq p$, then for every $n = 1, 2, \dots$

$$|\mu_n(E_i)| \leq |\mu_n(E_i)| \leq \sup_k |\mu_k(E_i)| < \infty.$$

For $i > p$, write

$$E_i = (B_0 \cup F_i) \setminus (B_0 \setminus F_i).$$

$$\text{Then } \lambda(B_0 \Delta (B_0 \cup F_i)) = \lambda(F_i \setminus B_0) \leq \lambda(F_i) \leq \lambda(E_i) \leq \epsilon,$$

$$\text{and } \lambda(B_0 \Delta (B_0 \setminus F_i)) = \lambda(B_0 \cap F_i) \leq \lambda(F_i) \leq \lambda(E_i) \leq \epsilon,$$

and so for all $n = 1, 2, \dots$,

$$|\mu_n(E_i)| \leq |\mu_n(B_0 \cup F_i)| + |\mu_n(B_0 \setminus F_i)| \leq 2m_0.$$

Therefore, for all n

$$|\mu_n(E)| \leq \sum_{i=p+1}^m |\mu_n(E_i)| + \sum_{i=1}^p |\mu_n(E_i)|.$$

$$\leq 2m_0(n-p) + p \max_{1 \leq i \leq p} \sup_k |\mu_k(E_i)|.$$

Since the right hand side of this inequality is finite and independent of n and E , the set $\{\mu_n\}$ is uniformly bounded.

3.2.11. THEOREM. A sequence $\{\mu_n\}$ in $\text{ca}(\Sigma)$ converges weakly to μ in $\text{ca}(\Sigma)$ if and only if it converges elementwise to μ .

PROOF. (\Rightarrow) since $v = v(E)$ is in $\text{ca}(\Sigma)^*$ for every E in Σ .

(\Leftarrow). Suppose $\{\mu_n\} \subset \text{ca}(\Sigma)$ is such that

$\mu_n(E) \rightarrow \mu(E)$ for every E in Σ . By 3.2.10,

$\{\mu_n\}$ is uniformly bounded and hence bounded in the

variation norm by 3.1.10. Let

$$\lambda = \sum_{n=1}^{\infty} \frac{|\mu_n|}{2^n}.$$

For every n there exists f_n in $L^1(\lambda)$ such that

$\mu_n(E) = \int_E f_n d\mu$, $\forall E \in \Sigma$. Since also $\mu < \lambda$,

there exists f_0 in $L^1(\lambda)$ such that $\mu(E) = \int_E f_0 d\mu$,

$\forall E \in \Sigma$. note that by 3.1.11, $\{f_n\}_{n \in \mathbb{N}}$ is uniformly

bounded in $L^1(\lambda)$, say $\|f_n\|_1 \leq M$, $\forall n \geq 0$.

Since $\mu_n(E) \rightarrow \mu(E)$ for all E , $\int_E f_n g d\lambda \rightarrow \int_E f_0 g d\lambda$

for every simple function. Since the simple functions are

L^∞ -dense in $L^1(\lambda)$, $\int_E f_n h d\lambda \rightarrow \int_E f_0 h d\lambda$ for every

μ_n in $L^{\infty}(\lambda) = L^1(\lambda)^*$. Hence $f_n \rightarrow f_0$ weakly in $L^1(\lambda)$, so $\mu_n \rightarrow \mu$ weakly in $ca(\Sigma)$ by 3.1.11..

3.2.12. COROLLARY. A sequence (f_n) in $L^1(\lambda)$ converges weakly to f in $L^1(\lambda)$ if and only if $\int_E f_n d\lambda \rightarrow \int_E f d\lambda$ for all E in Σ .

3.2.13. A Banach space X is said to be weakly complete provided every sequence (x_n) in X for which $(x^*(x_n))$ converges, $\forall x^* \in X^*$, (x_n) converges weakly in X .

3.2.14. THEOREM. $ca(\Sigma)$ is weakly complete. If λ is in $ca^+(\Sigma)$, then $L^1(\lambda)$ is weakly complete.

PROOF. Suppose (μ_n) in $ca(\Sigma)$ is such that $\gamma(\mu_n)$ converges for all γ in $ca(\Sigma)^*$. Then $(\mu_n(\Sigma))$ converges for every E in Σ . By 3.2.9 the limit function μ is in $ca(\Sigma)$, and by 3.2.11, $\mu_n \rightarrow \mu$ weakly.

The second statement follows from considering $L^1(\lambda)$ as a closed subspace of $ca(\Sigma)$.

3.2-15. LEMMA. Let A be a subalgebra of Σ with $\Sigma = \Sigma(A)$. Let $\{\cdot_n\}$ be a uniformly countable additive sequence on Σ to \mathbb{R} , and suppose $\lim \mu_n(E)$ exists for every E in A . Then $\lim \mu_n(E)$ exists for every E in Σ .

PROOF. Let $M = \{E \in \Sigma : \lim_n \mu_n(E) \text{ exists}\}$. Then

$M \supset A$ and we show M is a monotone class; hence $M = \Sigma$ by 1.1.5.

Suppose $E_n \in M$ and $E_n \uparrow E$ monotonically.

Then $\mu_n(E_n) \equiv \mu_n(E)$ uniformly over n (by the uniformly countable additivity), and so given $\epsilon > 0$ there exists N such that

$$||\mu_n(E_n) - \mu_n(E)|| < \frac{\epsilon}{3}, \forall n.$$

Since $\{\mu_n(E)\}_{n=1}^{\infty}$ converges, there exists N

such that

$$p, q \geq N \Rightarrow ||\mu_p(E_p) - \mu_q(E_q)|| < \frac{\epsilon}{3}.$$

Hence

$$p, q \geq N \Rightarrow ||\mu_p(E) - \mu_q(E)|| < \epsilon.$$

That is, $\{\mu_n(E)\}_1^{\infty}$ converges.

3.2.16 THEOREM. If K is a subset of $ca(\Sigma)$, the following three statements are equivalent.

(1) K is conditionally weakly compact.

(2) K is bounded and uniformly countably additive.

(3) K is bounded and there exists λ in $ca^+(\Sigma)$ such that K is uniformly absolutely continuous with respect to λ .

PROOF. (1) \Rightarrow (3). Suppose K is conditionally weakly compact, and suppose $\|\mu\| \leq M$, $\forall \mu \in K$ (K is bounded by the uniform boundedness principle.) We first show that for all $\epsilon > 0$ there exists a finite set

$\{\mu_1, \dots, \mu_n\} \subset K$ and $\delta > 0$ such that

$$|\mu_i|(E) < \delta \quad (i = 1, \dots, n) \Rightarrow |\mu(E)| < \epsilon, \forall \mu \in K.$$

Suppose this is not the case. Then there exists $\epsilon > 0$ such that no such finite set and δ exist. Choose any μ in K . Then there exist E_1 in Σ and μ_1 in K such that

such that

$$|\mu_1|(E_1) < \frac{1}{2} \text{ and } |\mu_1(E_1)| \geq \epsilon.$$

Next, choose E_2 in Σ and μ_2 in K such that

$$|\mu_2|(E_2) < \frac{1}{4} \quad (i = 1, 2) \text{ and } |\mu_2(E_2)| \geq \epsilon.$$

By induction there are sequences $\{\mu_n\} \subset K$ and

$\{E_n\} \subset \Sigma$ such that

$$|\mu_i|(E_n) < \frac{1}{2^n} \quad (i = 1, 2, \dots, n)$$

and

$$|\mu_{n+1}(E_n)| \geq \epsilon.$$

Since K is conditionally weakly compact, by the Eberlein-Smulian theorem we may assume $\mu_n \rightarrow \mu$ weakly

in $c_0(\Sigma)$ for some μ . Letting $\lambda_0 = \sum \frac{1}{2^n} |\mu_n|$,

by the Vitali-Hahn-Saks theorem $\{\mu_n\}$ is uniformly absolutely continuous with respect to λ_0 . But note

that

$$\lambda_0(E_n) = \sum_{m=1}^{\infty} \frac{1}{2^m} |\mu_m|(E_n)$$

$$< \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{2^m} + \sum_{n=m+1}^{\infty} \frac{1}{2^n}$$

$$\leq \frac{1}{2^m} + \sum_{n=m+1}^{\infty} \frac{1}{2^n}$$

$$\rightarrow 0 \text{ as } m \rightarrow \infty,$$

and so $\lim_{m \rightarrow \infty} \mu_n(E_n) = 0$ uniformly in n . This

contradicts $|\mu_{n+1}(E_n)| \geq \epsilon, \forall n$, and so the

desired conclusion holds.

It now follows that there exists a sequence

$\{v_n\} \subset K$ such that

$$|v_n|(E) = 0, \forall n \Rightarrow |\mu(E)| = 0, \forall \mu \in K.$$

Let $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} |v_n|$. We have $\mu < < \lambda, \forall \mu \in K$.

If we do not have uniform absolute continuity, then

there exist $\epsilon > 0$, a sequence $\{E_n\} \subset E$, and a

sequence $\{\mu_n\} \subset K$ such that $\lambda(E_n) \neq 0$ and

$|\mu_n(E_n)| \geq \epsilon, \text{ all } n$. Again, since K is con-

ditionally weakly compact, we can assume $\mu_n \rightarrow \mu$

weakly for some μ . This would say, by the Vitali-

Zahn-Saks theorem, that $\{\mu_n\}$ was uniformly absolutely

continuous, a contradiction. Hence (3) holds.

(3) \Rightarrow (2) clearly.

(2) \Rightarrow (1). Suppose K is bounded and uniformly countably additive. Let $\{\mu_n\}$ be a sequence in K ,

and let $\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|$. For each n , choose f_n

in $L^1(\lambda)$ such that

$$\mu_n(E) = \int_E f_n d\lambda, \quad \forall E \in \Sigma.$$

Since each f_n is the pointwise limit of a sequence of simple functions, there is a countable collection $E_n \subset \Sigma$ such that f_n is $\Sigma(E_n)$ -measurable. Let

$E = \bigcup_{n=1}^{\infty} E_n$, and let $A = A(E)$. Then E and hence

A is countable (by 1.1.3). Using the Cantor diagonal process, there is a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$

which converges elementwise on A . By 3.2.15,

$\{\mu_{n_k}\}$ converges elementwise on $\Sigma(A)$; by 3.2.12

$\{f_{n_k}\}$ converges weakly in $L^1(\lambda|\Sigma(A))$. Hence

$\{f_{n_k}\}$ converges weakly in $L^1(\lambda)$ since $L^1(\lambda|\Sigma(A))$

can be embedded isometrically into $L^1(\lambda)$. Finally,

$\{\mu_{n_k}\}$ converges weakly in $ca(\Sigma, \lambda) \subset ca(\Sigma)$ by the

Radon-Nikodym theorem.

By the Eberlein-Smulian theorem, Σ is conditionally weakly compact.

3.2.17. COROLLARY. If λ is in $ca^+(\Sigma)$ and K is a subset of $L^1(\lambda)$, then the following three statements are equivalent.

3.2-20

- (1) K is conditionally weakly compact in $L^k(\lambda)$.
- (2) K is bounded and the indefinite integrals of the members of K are uniformly countably additive.
- (3) K is bounded and the indefinite integrals of the members of K are uniformly absolutely continuous with respect to λ .

3.2.17

3.3.1

3.3 CONTROL MEASURES

3.3.1 If K is a subset of $ca(\Sigma)$, a control measure for K is a measure λ in $ca(\Sigma)$ such that every member of K is absolutely continuous with respect to $|\lambda|$.

A uniform control measure for K is a measure λ such that the members of K are uniformly absolutely continuous with respect to $|\lambda|$.

Theorem 3.2.16 tells us that if K is a bounded subset of $ca(\Sigma)$, then K is conditionally weakly compact if and only if there exists a uniform control measure for K . Moreover, if K is conditionally weakly compact, then every control measure is a uniform control measure.

3.3.2 LEMMA. Let μ and ν be in $ca(\Sigma)$. Then there exists a countable subset I of the reals \mathbb{R} such that for all s in $\mathbb{R} \setminus I$,

$$s\mu + (1-s)\nu$$

is a control measure for the set $K = \{\mu, \nu\}$.

PROOF. Let $\lambda = |\mu| + |\nu|$, and choose f, g in $L^1(\lambda)$ such that $\mu(E) = \int_E f d\lambda$ and $\nu(E) = \int_E g d\lambda$, ($E \in \Sigma$).

For every a in \mathbb{R} , let

$$E_a = \{\omega: f(\omega) + ag(\omega) = 0\} \cap (\{\omega: g(\omega) \neq 0\} \cup \{\omega: f(\omega) \neq 0\})$$

If $a \neq 0$, then $E_a \cap E_0 = \emptyset$, and so for all a except in a countable set J , $f(\omega) + ag(\omega)$ is non-zero for λ -almost all ω in the set

$$\{\omega: g(\omega) \neq 0\} \cup \{\omega: f(\omega) \neq 0\}.$$

It follows that for all a in $\mathbb{R} \setminus J$, $u + av$ is a control measure for the set $\{\mu, v\}$. Then $\beta u + (1 - \beta)v$ is a control measure for all β in the set

$$\{(\alpha + 1)^{-1} : \alpha \in \mathbb{R} \setminus J, \alpha \neq -1\}.$$

3.3.3. THEOREM. If K is any non-void subset of $ca(\Sigma)$, then the following four statements are equivalent.

(i) K has a control measure in $ca(\Sigma)$.

(ii) There is a sequence (μ_n) in K such that

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\mu_n\|}{\|\mu_n\| + 1}$$

is a control measure for K .

(iii) K has a control measure in its closed convex hull $\overline{co}(K)$.

(iv) The members of $\overline{\text{co}}(K)$ which are control measures for K form a dense subset of $\overline{\text{co}}(K)$. (In fact, for every v in $\overline{\text{co}}(K)$ there is a line L through v such that $L \cap \{\sigma : \sigma \text{ is a control measure for } K\}$ is dense in $L \cap \overline{\text{co}}(K)$).

PROOF. (i) \Rightarrow (ii). Let λ be a control measure for K . If $\mu \neq 0$ is in K and $\mu = fd|\lambda|$, then the set $A = \{x : f(x) \neq 0\}$ has the property that there exists some σ in K such that $|\sigma|(A) > 0$ and $|\lambda| \ll |\sigma|$ on A (in fact; take $\sigma = \mu$ here.) Use Zorn's lemma to choose a maximal disjoint family $\{A_i\}_{i \in I}$ of sets with this property, with corresponding measures $(\sigma_i)_{i \in I}$. Since $|\lambda|(A_i) > 0$ for each i , I must be countable. By maximality, there cannot exist an A in Σ with $|\sigma_i|(A) = 0$ for all i but $|\sigma|(A) > 0$ for some σ in K ; hence $(\sigma_i)_{i=1}^{\infty}$ is the desired sequence in K .

(ii) \Rightarrow (iii). For the proof we may assume K itself is closed and convex.

First we observe that we may assume K is bounded. Let μ be in K and let $B_1(\mu)$ denote the closed ball of radius 1 centered at μ . Suppose we can find an element σ in $B_1(\mu) \cap K$ which is a control measure for $B_1(\mu) \cap K$, and let v be any element of K . There exist $0 < a < 1$ such

that $\alpha u + (1 - \alpha)v$ is in $B_1(u) \cap K$. If $|\sigma|(A) = 0$, then $\mu(E) = 0$ and $[\alpha u + (1 - \alpha)v](E) = 0$, so $v(E) = 0$. Hence α is a control measure for all of K .

Thus we assume K is closed, bounded and convex. Let $\{u_n\}$ and λ be as in (ii).

Let $v_1 = u_1$, and by induction let v_k be any member of K which is a control measure for (u_{k-1}, u_k) . Then $|u_k| \ll |v_k|$ for all k , and

$$(1) \quad |v_1| \ll |v_2| \ll |v_3| \ll \dots$$

For each k , choose f_k in $L^1(\lambda)$ such that $v_k(E) = \int_E f_k d\lambda$, $(E \in \Sigma)$, and let $E_k = \{\omega: f_k(\omega) = 0\}$. By (1), we may assume (E_k) is a monotone decreasing sequence. Let $E = \bigcap_{k=1}^{\infty} E_k$. Then $|v_k|(E) = 0$ for all k , so $\lambda(E) = 0$; i.e., $\lambda(E_k) \rightarrow 0$. By passing to a subsequence, we may assume that

$$(2) \quad \lambda(\omega: f_n(\omega) = 0) < \frac{1}{2} \cdot \frac{1}{2^n}, \quad \forall n$$

We next define inductively two sequences $\{\alpha_n\}$, $\{\delta_n\}$ of positive real numbers. Let $\alpha_1 = 1$, and choose $\delta_1 > 0$ such that

$$\lambda(\omega: |f_1(\omega)| < \delta_1) < \frac{1}{2}.$$

Let $g_1 = f_1 (= \alpha_1 f_1)$. Now suppose $\alpha_1, \dots, \alpha_{n-1}$ and $\delta_1, \dots, \delta_{n-1}$ have been chosen, and let $\epsilon_{n-1} = \alpha_1 f_1 + \dots + \alpha_{n-1} f_{n-1}$.

Using the fact that f_n is in L^1 and the first part of the proof of the lemma, one sees that we may choose a_n such that

$$(3) \quad 0 < a_n \leq \frac{1}{2^{n-1}},$$

(4) $g_n = g_{n-1} + a_n f_n$ is non-zero λ -almost everywhere on the set

$$\{\omega: g_{n-1}(\omega) \neq 0\} \cup \{\omega: f_n(\omega) \neq 0\}$$

and

$$(5) \quad \lambda(\omega: |a_n f_n(\omega)| \geq \frac{\delta_{n-1}}{4}) < \frac{1}{2^n}.$$

By (2) and (4), we may choose δ_n such that

$$(6) \quad 0 < \delta_n < \frac{\delta_{n-1}}{4}$$

and

$$\lambda(\omega: |g_n(\omega)| \leq \delta_n) < \frac{1}{2^n}.$$

By (3), $\sum_{n=1}^{\infty} a_n < \infty$ and it follows that

$\sum_{n=1}^{\infty} |a_n f_n|$ converges in mean and hence λ -almost everywhere
(since it is a non-negative series.)

Hence

$$g = \sum_{n=1}^{\infty} a_n f_n = \lim_n g_n$$

converges in mean and λ -almost everywhere.

We next show that if $G = \{\omega : g(\omega) = 0\}$

$\lambda(G) = 0$. Suppose $\lambda(G) = a > 0$. Let

$$G_n = \{\omega : |\epsilon_n(\omega)| > \delta_n\}$$

and

$$F_n = \{\omega : |\alpha_n f_n(\omega)| \geq \frac{\delta_{n-1}}{4}\}.$$

By (7), $\lambda(G \cap G_n)$ converges to $\lambda(G) = a$; and by

(5), $\lambda(F_n) < \frac{1}{2^n}, \forall n$. Choose N so large that

$\lambda(G \cap G_N) > \frac{a}{2}$ and $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{a}{3}$. Then

$$\lambda[(G \cap G_N) \setminus \bigcup_{n=N+1}^{\infty} F_n] > \frac{a}{3},$$

so that in particular there exists a point x in

$(G \cap G_N) \setminus \bigcup_{n=N+1}^{\infty} F_n$ such that $g(x) = \sum_{n=1}^{\infty} \alpha_n f_n(x)$.

Then

$$|g(x)| = \left| \epsilon_N(x) - \sum_{n=N+1}^{\infty} (-\alpha_n f_n(x)) \right|$$

$$\geq |\epsilon_N(x)| - \sum_{n=N+1}^{\infty} |\alpha_n f_n(x)|$$

$$> \delta_N - \sum_{i=1}^{\infty} \frac{\delta_N}{4^i} = \frac{2}{3} \delta_N > 0,$$

which contradicts the fact that x is in G .

Now let $\sigma = \sum_{n=1}^{\infty} \left(\frac{a_n}{\sum_{i=1}^{\infty} a_i} \right) v_n$. Clearly

σ is in K . By the Radon-Nikodym theorem,

$$\sigma(E) = \int_E \left(\sum_{i=1}^{\infty} a_i \right)^{-1} g d\lambda, \quad \forall E \in \Sigma.$$

Since $\lambda(\omega: g(\omega) = 0) = 0$, $\lambda \ll |\sigma|$, and so σ is a control measure for K .

(iii) \Rightarrow (iv). Let σ be any member of $\overline{\text{co}}(K)$ which is a control measure for K , and let μ be any member of $\overline{\text{co}}(K)$. By the lemma, a dense subset of the line joining σ and μ consists of control measures for (σ, μ) . Any measure which controls σ automatically controls all of K .

3.3.4 COROLLARY. If K is a conditionally weakly compact subset of $\text{ca}(\mathbb{Z})$, then there exists a uniform control measure σ in $\overline{\text{co}}(K)$. In fact, the set of uniform control measures in $\overline{\text{co}}(K)$ is dense in $\overline{\text{co}}(K)$.

3.3.5 EXERCISE. Consider $\text{ca}(\mathbb{N})$, where \mathbb{N} is the set of natural numbers, and let $I = (g_n)_1^{\infty}$ be any countable subset of $(0,1)$. Let (a_n) be a positive sequence such that

$$\sum_{n=1}^{\infty} a_n |(s_n - 1)^{-1} s_n| < \infty$$

and

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Let μ and v be the members of $ca(\mathbb{Z}^N)$ such that
 $\mu(\{n\}) = a_n$ and $v(\{n\}) = a_n(s_n - 1)^{-1} s_n$. Then
 $\rho\mu + (1 - \rho)v$ is a control measure for $K = \{\mu, v\}$
if and only if s is not in I .

R.3.1. REMARKS AND REFERENCES

The most important result in this section is of course the classical Radon-Nikodym Theorem (3.1.9, 3.1.11). Considerable attention will be given to generalizations later (Chapters 7 and 8).

R.3.2. REMARKS AND REFERENCES

- (1) For a history of and generalizations of Theorem 3.2.7, see Pettis [1951(a)]. The entire power of 3.2.7 is of course not needed for 3.2.8, and a direct proof of 3.2.8 is shorter (see e.g., Dunford and Schwartz [1958, p. 158].)
- (2) Generalizing previous results of Vitali and Hahn, Saks [1933] proved the important Theorem 3.2.8. Corollary 3.2.9 was proved independently by Nikodym [1933(a)]; it extends to the vector-valued case (see 5.1.7 below.) Both 3.2.8 and 3.2.9 are generalized later (5.1.11 and 5.1.13).
- (3) Theorem 3.2.10 is due to Nikodym [1933]; the proof given here is due to Saks [1933]. This result is generalized later (5.1.10).
- (4) Theorems 3.2.11 and 3.2.12 go back to Lebesgue [1909, p. 57].
- (5) Theorems 3.2.16 and 3.2.17 have a long history culminating in the important paper by Bartle, Dunford, and Schwartz [1955]. The papers by Dubrovskii [1947(a), (b)] contain early partial results. The proofs given here are basically from Dunford and Schwartz [1958]. These results are generalized later (4.4.4).

Additional References: Ando [1961], Brooks [1969(a)], Gaussler [1971], Gould [1965], Hoffmann-Jørgensen [1971], Mikusinski [1970], Labuda [1972], Drewnowski [1972(a)].

R.3.3. REMARKS AND REFERENCES.

Bartle, Dunford, and Schwartz [1955] showed that if μ is in $ca(\Sigma, \mathbb{X})$, then the set $K = \{x^* \circ \mu: x^* \in X^*, \|x^*\| \leq 1\}$ has a uniform control measure in $ca(\Sigma)$. Rybakov [1970] showed that in this case, K contained a control measure for itself. Walsh [1971], using different methods, proved a version of 3.3.3. The proof given here of 3.3.3, using basically the methods of Rybakov, is due to Huff and Morris [1972].

Additional References: Drewnowski [1972(a)].