# 2. THE THEORY OF A REAL, NON-NEGATIVE, c.a. MEASURE ON A $\sigma$-ALGEBRA

Throughout this chapter, $\Sigma$ denotes a fixed $\sigma$-algebra of subsets of some set $\Omega$, and $\lambda$ is a fixed, real-valued, non-negative, c.a. measure on $\Sigma$. In particular, $\lambda$ is bounded.

## 2.1. MEASURABLE FUNCTIONS

2.1.1. Recall that a function $f:\Omega \to R$ is $(\Sigma)$-<u>measurable</u> if and only if $f^{-1}(I)$ is in $\Sigma$ for every interval $I \subset R$. The set $M(\Sigma)$ of all measurable functions is a linear space which is a lattice under pointwise suprema and infima, and is closed under pointwise limits of sequences.

Define $n:M(\Sigma) \to R$ by

$$n(f) = \inf_{\varepsilon>0}[\varepsilon + \lambda\{w \in \Omega : |f(w)| \geq \varepsilon\} \quad (f \in M(\Sigma)).$$

The proofs of the following properties of $n$ are left as easy exercises:

(i) $n(f) \geq 0$, $n(f) = 0$ iff $f = 0$ a.e. $[\lambda]$,

(ii) $n(-f) = n(f)$

(iii) $n(f+g) \leq n(f) + n(g)$.

Thus $n$ is almost a norm on $M(\Sigma)$; using it, we define a pseudo-metric $\rho$ on $M(\Sigma)$ by

$$\rho(f,g) = n(f-g) \qquad (f,g \in M(\Sigma)).$$

Then $\rho$ is a translation-invariant pseudo-metric on $M(\Sigma)$, i.e., for all $f,g,h$ in $M(\Sigma)$

(a) $\rho(f,g) \geq 0$, $\rho(f,f) = 0$

(b) $\rho(f,g) = \rho(g,h)$

(c) $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$

and (d) $\rho(f+h,g+h) = \rho(f,g)$.

A function $f$ in $M(\Sigma)$ is called a $\lambda$-<u>null function</u> provided $n(f) = 0$, i.e., $f = 0$ a.e. $[\lambda]$. Letting $N(\lambda)$ denote the $\lambda$-null functions in $M(\Sigma)$, $\rho$ is a metric on the quotient space $M(\Sigma)/N(\lambda)$.

2.1.2. THEOREM. <u>A sequence</u> $\{f_n\}$ <u>in</u> $M(\Sigma)$ $\rho$-<u>converges to an element</u> $f$ <u>in</u> $M(\Sigma)$ <u>if and only if for every</u> $\varepsilon > 0$

(#) $$\lim_{n\to\infty} \lambda\{w : |f_n(w) - f(w)| \geq \varepsilon\} = 0.$$

PROOF. Since both $\rho$-convergence and the limit condition (#) are translation invariant, we need only consider the case when $f = 0$.

Suppose for every $\varepsilon > 0$, $\lambda\{w : |f_n(w)| \geq \varepsilon\} \overset{n}{\to} 0$. Given $\delta > 0$, let $\varepsilon = \delta/2$ and choose $n_0$ such that

$$n \geq n_0 \Rightarrow \lambda\{w: |f_n(w)| \geq \varepsilon\} < \delta/2.$$

Then

$$\begin{aligned} n \geq n_0 &\Rightarrow \varepsilon + \lambda\{w: |f_n(w)| \geq \varepsilon\} < \delta \\ &\Rightarrow \rho(f_n, 0) < \delta. \end{aligned}$$

Thus $f_n \to 0$ in the $\rho$-topology.

Conversely, suppose $f_n \to 0$ in the $\rho$-topology. Let $\varepsilon > 0$ be given. Given any $\delta$ with $0 < \delta < \varepsilon$, there exists $n_0$ such that

$$n \geq n_0 \Rightarrow \inf_{\beta > 0} [\beta + \lambda\{w : |f_n(w)| \geq \beta\}] < \delta.$$

Suppose $\lambda\{w : |f_n(w)| \geq \varepsilon\} \geq \delta$ for some $n \geq n_0$. Then

$$\beta + \lambda\{w: |f_n(w)| \geq \beta\} \begin{cases} \geq \beta \geq \varepsilon \geq \delta & \text{if } \beta \geq \varepsilon \\ \geq \lambda\{w: |f_n(w)| \geq \varepsilon\} \geq \delta & \text{if } \beta < \varepsilon, \end{cases}$$

a contraction. Hence $n \geq n_0 \Rightarrow \lambda\{w: |f_n(w)| \geq \varepsilon\} < \delta$, so $f_n$ converges to $0$ in the sense of (#).

2.1.3. EXERCISES. (1) Show that a sequence $\{f_n\}_1^\infty \subset M(\Sigma)$ is $\rho$-Cauchy if and only if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \lambda\{w: |f_n(w) - f_m(w)| \geq \varepsilon\} = 0.$$

(2) Show that 2.1.2 holds if the sequence $\{f_n\}_1^\infty$ is replaced by an arbitrary net $\{f_\alpha\}_{\alpha\in D}$.

2.1.4. The $\rho$-topology on $M(\Sigma)$ is called <u>the topology of convergence in $\lambda$-measure</u>. It is easily checked that the following functions are continuous with respect to this topology:

(i) $a: M(\Sigma) \times M(\Sigma) \to M(\Sigma)$, where $a(f,g) = f + g$.

(ii) $m_\alpha: M(\Sigma) \to M(\Sigma)$, where $m_\alpha(f) = \alpha f$, where $\alpha$ is an arbitrary but fixed real number.

(iii) $v: M(\Sigma) \to M(\Sigma)$, where $v(f) = |f|$.

2.1.5. EXERCISE. Let $\lambda$ be defined on $\Sigma = 2^R$ by

$$\lambda(A) = \varepsilon_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A. \end{cases}$$

Show that

$$n(f) = \begin{cases} 1 & \text{if } f(0) \neq 0 \\ 0 & \text{if } f(0) = 0 , \end{cases}$$

so $n$ is <u>not</u> a pseudo-norm on $M(\Sigma)$ (since $n(\alpha f) \neq |\alpha| n(f)$.) Conclude also that the $\rho$-topology on $M(\Sigma)$ is not a linear topology; i.e., with it $M(\Sigma)$ does <u>not</u> form a linear topological space.

2.1.6. THEOREM (Egoroff's Theorem). <u>If</u> $f_n$, $f$ <u>are in</u> $M(\Sigma)$, <u>then</u> $f_n \to f$ <u>a.e.</u> $[\lambda]$ <u>if and only if</u>

$$(\#) \quad \forall \varepsilon > 0 \ \exists A \in \Sigma \ni \lambda(A) < \varepsilon \ \underline{\text{and}} \ f_n \to f \ \underline{\text{uniformly on}} \ \Omega \setminus A.$$

PROOF. It is trivial to show $(\#) \Rightarrow f_n \to f$ a.e. $[\lambda]$.

Conversely, assume $f_n \to f$ a.e. $[\lambda]$. Then by translation, assume $f = 0$ and by redefining the functions on a set of measure zero, assume $f_n \to 0$ everywhere. For all $k,m = 1,2,\ldots,$ let

$$E_{k,m} = \{\omega : |f_n(\omega)| < \frac{1}{m} , \ \forall n \geq k\}.$$

Then $E_{k+1,m} \supset E_{k,m}$ and $\Omega = \bigcup_{k=1}^{\infty} E_{k,m}$. Hence

$$\lambda(E_{k,m}) \nearrow \lambda(\Omega) \quad \text{as } k \to \infty \text{ (fixed } m).$$

Given $\varepsilon > 0$, for each $m$ choose $k_m$ such that

$$\lambda(\Omega \setminus E_{k_m,m}) < \frac{\varepsilon}{2^m}.$$

Let $A = \bigcup_{m=1}^{\infty} (\Omega \setminus E_{k_m,m})$. Then $\lambda(A) < \varepsilon$ and for all $\omega$ in $\Omega\setminus A = \bigcap_{m=1}^{\infty} E_{k_m,m}$,

$$|f_n(\omega)| < \frac{1}{m} \quad \text{whenever} \quad n \geq k_m.$$

Hence $f_n \to 0$ uniformly on $\Omega\setminus A$.

2.1.7. COROLLARY. <u>If</u> $f_n \to f$ a.e. $[\lambda]$, <u>then</u> $f_n \to f$ <u>in</u> $\lambda$-<u>measure</u>.

PROOF. If $f_n \to f$ a.e. $[\lambda]$, then by 2.1.6 given $\varepsilon > 0$ and $\delta > 0$ $\exists$ set $A \in \Sigma$ and an integer $N$ such that $\lambda(A) < \varepsilon$ and

$$n \geq N \Rightarrow \|f_n - f\|_\infty < \delta \quad \text{on} \quad \Omega \setminus A$$

$$\Rightarrow \lambda\{\omega : |f_n(\omega) - f(\omega)| \geq \delta\} \leq \lambda(A) < \varepsilon.$$

Hence for every $\delta > 0$, $\lambda\{\omega : |f_n(\omega) - f(\omega)| \geq \delta\} \to 0$.

2.1.8. THEOREM. <u>If</u> $\{f_n\}$ <u>is a</u> $\rho$-<u>Cauchy sequence in</u> $M(\Sigma)$, <u>then there exists a subsequence</u> $\{f_{n_k}\}$ <u>which converges a.e.</u> $[\lambda]$ <u>to some member of</u> $M(\Sigma)$.

PROOF. Since $\{f_n\}$ is Cauchy in measure, we can choose $n_1 < n_2 < n_3 < \ldots$ such that for every $k$

$$m,n \geq n_k \Rightarrow \lambda\{\omega: |f_n(\omega)-f_m(\omega)| \geq \frac{1}{2^k}\} < \frac{1}{2^k}.$$

Consider the subsequence $\{f_{n_k}\}$ of $\{f_n\}$. Let

$$E_k = \{\omega: |f_{n_k}(\omega)-f_{n_{k+1}}(\omega)| \geq \frac{1}{2^k}\}.$$

By construction $\lambda(E_k) < \frac{1}{2^k}$. Let $A = \bigcap_{k=1}^{\infty} (\bigcup_{\ell=k}^{\infty} E_\ell)$. Then $\lambda(A) \leq \lambda(\bigcup_{\ell=k}^{\infty} E_\ell) < \frac{1}{2^{k-1}}$ for every $k$, so $\lambda(A) = 0$. Now suppose $\omega_0 \in \Omega\backslash A$, and let $\varepsilon > 0$ be given. There exists $k$ such that $\frac{1}{2^{k-1}} < \varepsilon$ and such that $\omega_0 \notin \bigcup_{\ell=k}^{\infty} E_\ell$. Thus

$$\ell \geq k \Rightarrow \omega_0 \notin \{\omega: |f_{n_\ell}(\omega)-f_{n_{\ell+1}}(\omega)| \geq \frac{1}{2^\ell}\}$$

$$\Rightarrow |f_{n_\ell}(\omega_0)-f_{n_{\ell+1}}(\omega_0)| < \frac{1}{2^\ell},$$

and so

$$\ell \geq k \Rightarrow \forall p, \quad |f_{n_\ell}(\omega_0)-f_{n_{\ell+p}}(\omega_0)| < \frac{1}{2^{k-1}} \quad .$$

Hence $\{f_{n_\ell}\}$ converges pointwise on $\Omega\backslash A$, so $f_{n_k} \to f$ a.e. $[\lambda]$ for some $f \in M(\Sigma)$.

2.1.9 COROLLARY. $M(\Sigma)$ <u>is a complete pseudometric space in the topology of convergence in measure</u>.

PROOF. Suppose $\{f_n\}$ is $\rho$-Cauchy. Then some subsequence $\rho$-converges to some $f$, and hence $\{f_n\}$ $\rho$-converges to $f$.

2.1.10 Recall that a ($\Sigma$-) <u>simple function</u> is a function of the form

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i} ,$$

where $\alpha_i \in R$, $E_i \in \Sigma$ $(i = 1,\ldots,n)$. Such a function has a unique representation of this form if we assume $i \neq j \Rightarrow \alpha_i \neq \alpha_j$ and $E_i \cap E_j = \emptyset$ and that $\Omega = \bigcup_{i=1}^{n} E_i$. The simple functions form a linear subspace of $M(\Sigma)$. Every member $f$ of $M(\Sigma)$ is the pointwise limit of a sequence $\{f_n\}$ of simple functions; if $f \geq 0$ we may take $f_n \geq 0$, $\forall_n$; if $f$ is bounded we may choose the $f_n$'s such that $f_n \to f$ uniformly.

2.1.11 EXERCISE. Find a sequence of functions $f_n$ such that $f_n \to 0$ in measure but such that $f_n$ does <u>not</u> converge a.e. $[\lambda]$. Thus convergence in measure and convergence a.e. $[\lambda]$ do <u>not</u> coincide.

Note that $f_n \to 0$ in measure implies that every subsequence $\{f_{n_k}\}$ has a sub-subsequence which converges a.e. to 0. Conclude that a.e. convergence is <u>not</u> a topological mode of convergence; i.e., there is no topology on $M(\Sigma)$ for which convergence of sequences is precisely convergence almost everywhere $[\lambda]$.

## 2.2 MEASURABLE SETS

2.2.1 The function $E \to \chi_E$ is a one-to-one map of $\Sigma$ into $M(\Sigma)$; its range is the set of all $\{0,1\}$-valued members of $M(\Sigma)$. If the pseudo-metric $\rho$ on $M(\Sigma)$ is "pulled back" to $\Sigma$ in the natural way, then by 2.1.9, $\Sigma$ becomes a complete pseudo-metric space. We have for $A,B$ in $\Sigma$

$$\begin{aligned}\overline{\rho}(A,B) &= \rho(\chi_A,\chi_B) \qquad \text{(definition)}\\ &= \inf_{\varepsilon>0} [\varepsilon + \lambda\{\omega: |\chi_A(\omega)-\chi_B(\omega)| \geq \varepsilon\}\\ &= \min(\lambda(A \triangle B),1),\end{aligned}$$

where $A \triangle B$ is the symmetric difference $(A\backslash B)\cup(B\backslash A)$. An equivalent pseudo-metric is

$$d(A,B) = \lambda(A \triangle B), \qquad (A,B \in \Sigma).$$

It is easily checked that the following four functions are all continuous with respect to $d$.

(i) $(A,B) \mapsto A \cap B \qquad ((A,B) \in \Sigma \times \Sigma)$

(ii) $A \mapsto \Omega\backslash A \qquad (A \in \Sigma)$

(iii) $(A,B) \mapsto A \cup B \qquad ((A,B) \in \Sigma \times \Sigma)$

(iv) $(A,B) \mapsto A\backslash B \qquad ((A,B) \in \Sigma \times \Sigma)$.

2.2.2 PROPOSITION. *If* $\mathcal{A}$ *is a sub-algebra of* $\Sigma$, *then its d-closure* $\overline{\mathcal{A}}$, *is*

$$\Sigma(\mathcal{A}) \,\Delta\, \{\text{sets of measure zero}\}$$
$$= \{A \,\Delta\, B : A \in \Sigma(\mathcal{A}),\ B \in \Sigma,\ \lambda(B) = 0\}.$$

PROOF. Restricting $\lambda$ to $\Sigma(\mathcal{A})$, $\Sigma(\mathcal{A})$ is complete and hence its closure is $\Sigma(\mathcal{A}) \,\Delta\, \{\text{sets of measure zero}\}$.

It follows from continuity of set operations that $\overline{\mathcal{A}}$ is an algebra. Suppose $\{E_i\}_{i=1}^{\infty}$ is in $\overline{\mathcal{A}}$. Then $F_n = \bigcup_{i=1}^{n} E_i$ is in $\overline{\mathcal{A}}$ and $F_n \nearrow (\bigcup_{i=1}^{\infty} E_i)$, so

$$\lambda\Big(\big(\bigcup_{i=1}^{\infty} E_i\big) \,\Delta\, F_n\Big) = \lambda\Big(\big(\bigcup_{i=1}^{\infty} E_i\big) \setminus F_n\Big) \xrightarrow{n} 0.$$

Hence $\bigcup_{i=1}^{\infty} E_i$ is in $\overline{\mathcal{A}}$.

2.2.3 COROLLARY. *The measure* $\lambda$ *is completely determined by its values on any sub-algebra* $\mathcal{A}$ *of* $\Sigma$ *for which* $\Sigma = \Sigma(\mathcal{A})$.

PROOF. Note that $\lambda$ is (uniformly) d-continuous on $\Sigma$ since $|\lambda(A)-\lambda(B)| = |\lambda(A\setminus B)-\lambda(B\setminus A)| \leq \lambda(A\setminus B) + \lambda(B\setminus A) = \lambda(A \,\Delta\, B) = d(A,B)$. By 2.2.2, if $\Sigma = \Sigma(\mathcal{A})$, then $\mathcal{A}$ is d-dense in $\Sigma$.

2.2.4 For $\lambda$ in $ca^{+}(\Sigma)$, the <u>measure algebra</u> $\Sigma/\lambda$ is the set of all equivalence classes

$$\overline{E} = \{F \in \Sigma : \lambda(E \,\Delta\, F) = 0\}, \quad (E \in \Sigma).$$

We carry $\lambda$ over to $\Sigma/\mu$ by letting $\overline{\lambda}(\overline{E}) = \lambda(E)$. This and the following operations are well-defined

$$\overline{E} \cap \overline{F} = \overline{(E \cap F)}$$

$$\overline{E} \cup \overline{F} = \overline{(E \cup F)}$$

$$\overline{E} \setminus \overline{F} = \overline{(E \setminus F)}$$

$$\overline{E} \,\Delta\, \overline{F} = \overline{(E \,\Delta\, F)}$$

$$\left(\bigcup_{n=1}^{\infty} \overline{E}_n\right) = \overline{\left(\bigcup_{n=1}^{\infty} E_n\right)}$$

$$\left(\bigcap_{n=1}^{\infty} \overline{E}_n\right) = \overline{\left(\bigcap_{n=1}^{\infty} E_n\right)}.$$

Write $\overline{E} \subset \overline{F}$ if and only if $\overline{E} \cap \overline{F} = \overline{E}$. From above we know that $\Sigma/\lambda$ is a complete metric space under the metric $\overline{d}(\overline{E},\overline{F}) = \overline{\lambda}(\overline{E} \,\Delta\, \overline{F})$.

2.2.5 A set $A \subset \Sigma$ is said to be an <u>atom</u> of $\lambda$ if $\lambda(A) \neq 0$ and

$$B \subset A,\ B \in \Sigma \Rightarrow \lambda(A) = \lambda(B) \text{ or } \lambda(B) = 0.$$

Equivalently, $A$ is an atom of $\lambda$ if and only if

$$\overline{B} \subset \overline{A},\ \overline{B} \in \Sigma/\lambda \Rightarrow \overline{B} = \overline{A} \text{ or } \overline{B} = \overline{\phi}.$$

The measure $\lambda$ is said to be <u>non-atomic</u> if there are no atoms of $\lambda$ in $\Sigma$; $\lambda$ is said to be <u>purely atomic</u> if $\Omega$ can be expressed as the union of atoms.

2.2.6 THEOREM. <u>If</u> $\lambda \in ca^+(\Sigma)$ <u>is non-atomic, then for every</u> $E$ <u>in</u> $\Sigma$ <u>with</u> $\lambda(E) > 0$,

$$\{\lambda(F): F \in \Sigma,\ F \subset E\} = [0, \lambda(E)].$$

PROOF. We first show that there exist subsets of $E$ with arbitarily small <u>positive</u> measure. Suppose this is not the case. Let

$$A = \{\overline{F} \in \Sigma/\lambda : \overline{F} \subset \overline{E},\ \overline{\lambda}(\overline{F}) > 0\}.$$

Order $A$ by inclusion, let $C$ be a chain in $A$, and let $\beta = \inf\{\overline{\lambda}(\overline{F}): \overline{F} \in C\}$. Note that by assumption $\beta > 0$. If there exists $\overline{F}$ in $C$ with $\overline{\lambda}(\overline{F}) = \beta$, then $\overline{F}$ is a lower bound for $C$. Suppose $\overline{\lambda}(\overline{F}) > \beta$ for every $\overline{F}$ in $C$, and choose a sequence $\{\overline{F}_n\}$ in $C$ with $\overline{\lambda}(\overline{F}_n) \searrow \beta$. Then since $C$ is a chain, $m \geq n \Rightarrow \overline{F}_m \subset \overline{F}_n$. Hence $\overline{\lambda}(\bigcap_{n=1}^{\infty} \overline{F}_n) = \beta$.

If $\overline{F}$ is in $\mathcal{C}$, then $\beta < \overline{\lambda}(\overline{F})$, so $\overline{\lambda}(\overline{F}_n) < \overline{\lambda}(\overline{F})$ for $m$ sufficiently large. Since $\mathcal{C}$ is a chain, $\overline{F} \supset \overline{F}_m \supset \bigcap_{n=1}^{\infty} \overline{F}_n$. Hence $\bigcap_{n=1}^{\infty} \overline{F}_n$ is a lower bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{A}$ has a minimal element $\overline{A}$. But then $A$ must be an atom, contrary to hypothesis. Therefore, if $\lambda(E) > 0$, then $E$ must contain sets with arbitrarily small positive measure.

Now let $0 < \alpha < \lambda(E)$ hold. By an argument similar to the above,

$$\mathcal{B} = \{\overline{F} \in \Sigma/\lambda : \overline{F} \subset \overline{E},\ \alpha \leq \overline{\lambda}(\overline{F})\}$$

has a minimal element, say $\overline{F}$. If $\overline{\lambda}(\overline{F}) > \alpha$, choose $\overline{G} \subset \overline{F}$ with $0 < \overline{\lambda}(\overline{G}) < \overline{\lambda}(\overline{F}) - \alpha$. Then $\overline{F} \setminus \overline{G}$ is in $\mathcal{B}$, contradicting minimality of $\overline{F}$ in $\mathcal{B}$. Thus there exists $F \subset E$ with $\lambda(F) = \alpha$.

2.2.7 COROLLARY. *If* $\lambda \in ca^{+}(\Sigma)$ *is non-atomic, if* $\alpha_i > 0$ $(i = 1,2,\ldots,n)$, *and if* $\sum_{i=1}^{n} \alpha_i = \lambda(\Omega)$, *then there exists a* $\Sigma$*-partition* $\pi = \{E_1,\ldots,E_n\}$ *of* $\Omega$ *such that* $\lambda(E_i) = \alpha_i$ *for all* $i = 1,\ldots,n$.

2.2.8 THEOREM. *If* $\lambda$ *is in* $ca^{+}(\Sigma)$, *then there exists an essentially unique decomposition of* $\Omega$ *into disjoint sets* $A, B \in \Sigma$ *such that* $A$ *is a union of a countable number of atoms of* $\lambda$ *and* $B$ *contains no atoms of* $\lambda$. *Thus, if* $\lambda_1(E) = \lambda(E \cap A)$ *and*

$\lambda_2(E) = \lambda(E \cap B)$, $(E \in \Sigma)$, __then__ $\lambda_1$ __is purely atomic__, $\lambda_2$ __is non-atomic__, __and__ $\lambda = \lambda_1 + \lambda_2$.

PROOF. Let $\{A_i\}_{i \in I}$ be a maximal disjoint family of atoms of $\lambda$. Since $\lambda(\Omega) < \infty$, $I$ is countable. Let $A = \bigcup_{i \in I} A_i$ and $B = \Omega \backslash A$.

2.2.9 Let $\Sigma$ and $\mathcal{M}$ be $\sigma$-algebras of subsets of $\Omega$ and $\Lambda$ respectively let $\lambda$ be in $ca^+(\Sigma)$, and let $m$ be in $ca^+(\mathcal{M})$. A function $\psi : \Sigma/\lambda \to \mathcal{M}/m$ is said to be an __isomorphism__ provided for all $E, F$ in $\Sigma$,

(i) $\psi(\overline{E} \cup \overline{F}) = \psi(\overline{E}) \cup \psi(\overline{F})$

(ii) $\psi(\overline{\Omega} \backslash \overline{E}) = \overline{\Lambda} \backslash \psi(\overline{E})$

and (iii) $\overline{m}(\psi(\overline{E})) = \overline{\lambda}(\overline{E})$.

We note that an isomorphism $\psi : \Sigma/\lambda \to \mathcal{M}/m$ has the following additional properties.

(iv) $\psi(\overline{E} \cap \overline{F}) = \psi(\overline{E}) \cap \psi(\overline{F})$

(v) $\psi(\overline{E} \backslash \overline{F}) = \psi(\overline{E}) \backslash \psi(\overline{F})$

(vi) $\psi(\overline{E} \,\Delta\, \overline{F}) = \psi(\overline{E}) \,\Delta\, \psi(\overline{F})$

(vii) $\psi$ is an isometry; i.e.,

$$\overline{m}(\psi(\overline{E}) \,\Delta\, \psi(\overline{F})) = \overline{\lambda}(\overline{E} \,\Delta\, \overline{F}).$$

(viii) $\psi(\bigcap_{n=1}^{\infty} \overline{E}_n) = \bigcup_{n=1}^{\infty} \psi(\overline{E}_n)$

and (ix) $\psi(\bigcap_{n=1}^{\infty} \overline{E}_n) = \bigcap_{n=1}^{\infty} \psi(\overline{E}_n)$.

PROOF. (iv) - (vii) are trivial. To prove (viii) we need only consider the case when the $E_n$'s are disjoint. Then $\psi(\bigcup_{n=1}^{\infty} \overline{E}_n) = \psi(\lim_m \bigcup_{n=1}^{m} \overline{E}_n) = \lim_m \bigcup_{n=1}^{m} \psi(\overline{E}_n) = \bigcup_{n=1}^{\infty} \psi(\overline{E}_n)$ by (i) and (vii). Statement (ix) follows by complementation.

2.2.10 THEOREM. Let $\lambda$ be a member of $ca^+(\Sigma)$ with $\lambda(\Omega) = 1$ and suppose $\Sigma/\lambda$ is separable. Let $M$ denote the σ-algebra of Borel subsets of $[0,1]$, and let $m$ denote Lebesgue measure on $M$. Then there exists an isomorphism $\psi:\Sigma/\lambda \to M/m$. If $\lambda$ is non-atomic, then $\psi$ can be taken to be onto $M/m$.

PROOF. Let $E = \{E_n\}_{n=1}^{\infty}$ be a sequence in $\Sigma$ such that $\{\overline{E}_n\}_{n=1}^{\infty}$ is dense in $\Sigma/\lambda$. For each $n$, let $\pi_n$ denote the collection of atoms of $\{E_1,\ldots,E_n\}$. Then each $\pi_n$ is a partition of $\Omega$, and $\pi_{n+1}$ refines $\pi_n$.

Note that $\pi_1 = \{E_1, \Omega \setminus E_1\}$. Let $\varphi(E_1) = [0,\lambda(E_1)]$, and let $\varphi(\Omega\setminus E_1) = (\lambda(E_1),1]$.

Now suppose $\varphi$ has been defined on $\pi_n = \{F_1,\ldots,F_m\}$ to $M$, say $\varphi(F_i) = I_i$, where $\{I_1,\ldots,I_m\}$ is a partition of $[0,1]$

into intervals such that $m(I_i) = \lambda(F_i)$, $i = 1,\ldots,m$. Let $\pi_{n+1} = \{G_1,\ldots,G_p\}$. Recall that $\pi_{n+1}$ refines $\pi_n$. We may assume $F_1 = \bigcup_{i=1}^{q} G_i$. Choose $\{J_1,\ldots,J_q\}$, a partition of $I_1$ into intervals such that $m(J_j) = \lambda(G_j)$, $j = 1,\ldots,q$. Let $\varphi(G_j) = J_j$, $j = 1,\ldots,q$. Define $\varphi$ on the rest of the members of $\pi_{n+1}$ similarly.

By induction, $\varphi$ is defined on all of $\bigcup_{n=1}^{\infty} \pi_n$. If $A$ is in $\mathcal{A}(E)$, then by 1.1.2 and 1.1.3, $A$ is the (disjoint) union of members of a partition $\pi_n$ for some $n$, say $A = \bigcup_{i\in I} F_i$, where the $F_i$'s are members of $\pi_n$. Let $\varphi(A) = \bigcup_{i\in I} \varphi(F_i)$.

We have now defined $\varphi:\mathcal{A}(E) \to \mathcal{M}$ such that it preserves the algebra operations and it preverves measure. Define $\overline{\varphi}:\overline{\mathcal{A}(E)} \to \mathcal{M}/m$, where $\overline{\mathcal{A}(E)} = \{\overline{E} \in \Sigma/\lambda : E \in \mathcal{A}(E)\}$, by

$$\overline{\varphi}(\overline{E}) = \overline{\varphi(E)} \qquad (E \in \mathcal{A}(E)).$$

Then $\overline{\varphi}$ is an isometry and hence extends uniquely to an isometry

$$\psi:\Sigma/\lambda \to \mathcal{M}/m$$

since $\overline{\mathcal{A}(E)}$ is dense in $\Sigma/\lambda$.

We verify that $\psi$ is an isomorphism. Let $E$ and $F$ be in $\Sigma$, and choose $\overline{E}_n$, $\overline{F}_n$ in $\overline{A(E)}$ such that $\overline{E}_n \to \overline{E}$ and $\overline{F}_n \to \overline{F}$. Then

$$\begin{aligned}\psi(\overline{E} \cup \overline{F}) &= \psi(\lim (\overline{E}_n \cup \overline{F}_n)) \\ &= \lim \psi(\overline{E}_n \cup \overline{F}_n) \\ &= \lim \psi(\overline{E}_n) \cup \psi(\overline{F}_n) \\ &= \psi(\overline{E}) \cup \psi(\overline{F}),\end{aligned}$$

since $\psi$ is an isometry. Similarly $\psi(\overline{\Omega \backslash E}) = \overline{[0,1]}\ \psi(\overline{E})$ and $m(\psi(\overline{E})) = \lambda(\overline{E})$.

Finally, suppose $\lambda$ is non-atomic. Let $I$ be any interval in $[0,1]$, and let $\varepsilon > 0$ be given. By 2.2.7, we can partition $\Omega$ into sets $F_1,\ldots,F_m$ such that $\lambda(F_i) < \varepsilon/4$ for each $i$. Choose $n$ so large that each $F_i$ is within $\varepsilon/4$ of a finite union of members of $\pi_n = \{G_1,\ldots,G_s\}$. Then $\max\limits_j \lambda(G_j) < \varepsilon/2$, and so $\{\varphi(G_1,\ldots,\varphi(G_s)\}$ is a partition of $[0,1]$ into intervals such that $\max\limits_j m(\varphi(G_j)) < \varepsilon/2$. Let $A = \bigcup\limits_{\varphi(G_j)\cap I \neq \emptyset} G_j$. Then $m(I \,\Delta\, \varphi(A)) < \varepsilon$. It follows that there exists a sequence $\{\overline{A}_n\}$ in $\Sigma/\lambda$ such that $\psi(\overline{A}_n) \to \overline{I}$ in $M/m$. Since $\psi$ is an isometry and $\Sigma/\lambda$ is complete, $\overline{A}_n \to \overline{A}$ for some $A$ in $\Sigma$. Hence $\overline{I} = \psi(\overline{A})$. This proves that $\psi(\Sigma/\lambda)$

includes $\{\overline{I}: I$ is an interval in $[0,1]\}$; it is also closed under countable unions and complements, and it follows that $\psi(\Sigma/\lambda)$ includes all of $M/m$.

2.2.11 If $\lambda \in ca^{+}(\Sigma)$ is purely atomic, then $\Sigma/\lambda$ is isomorphic to either $2^{N}/\nu$ or $2^{\{1,\ldots,n\}}/\nu$ for some $n$, where $\nu$ is some measure. Thus, it follows from 2.2.8 and 2.2.9 that whenever $\Sigma$ is countably generated and $\lambda$ is in $ca^{+}(\Sigma)$, $\Sigma/\lambda$ is isomorphic to one of

(a) the measure algebra of some interval $[0,a]$ with Lebesgue measure.

(b) the measure algebra $2^{J}/\nu$ for some $J \subset N$

and (c) a "direct sum" of (a) and (b).

2.2.12 THEOREM. <u>If</u> $\lambda$ <u>is in</u> $ca^{+}(\Sigma)$ <u>and</u> $\lambda$ <u>is not purely atomic, then there exist</u> $\varepsilon > 0$ <u>and a sequence</u> $\{E_n\}_1^{\infty}$ <u>in</u> $\Sigma$ <u>such that</u> $\lambda(E_n \Delta E_m) \geq \varepsilon$ <u>for</u> $n \neq m$. <u>In particular,</u> $\Sigma/\lambda$ <u>is not compact.</u>

PROOF. If $\lambda$ is not purely atomic, then by 2.2.8 we can find $A_{0_1}$ in $\Sigma$ such that $\lambda(A_{0_1}) > 0$ and $A_{0_1}$ contains no atoms. By an induction argument using 2.2.7 (or 2.2.6), we can find a doubly-indexed collection $\{A_{ni}: n = 0,1,2,\ldots;\ 1 \leq i \leq 2^{n}\}$ in $\Sigma$ such that for every $n$ and $i$, $\lambda(A_{ni}) = \frac{1}{2^{n}}\lambda(A_{01})$ and $A_{ni}$ is the disjoint union of $A_{n+1,2i-1}$ and $A_{n+1,2i}$. Let

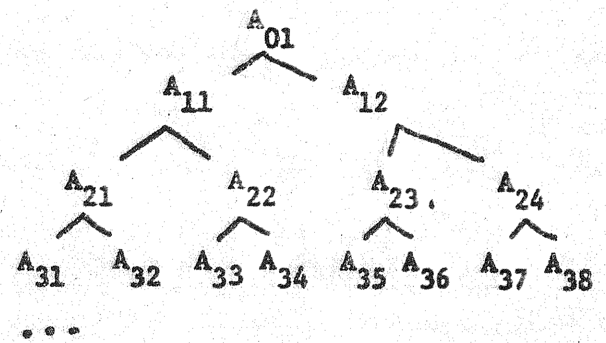
2.2.11

$E_1 = A_{11}$,

$E_2 = A_{21} \cup A_{23}$,

$E_3 = A_{31} \cup A_{33} \cup A_{35} \cup A_{37}$,

$\vdots$




Then for $n \neq m$, $\lambda(E_n \triangle E_m) = \frac{1}{2}\lambda(A_{01})$.

## 2.3 THE INTEGRAL

2.3.1. We let $\mathcal{S}(\Sigma)$ denote the space of all $\Sigma$-simple functions. If $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ is in $\mathcal{S}(\Sigma)$, its $\lambda$-integral is defined by

$$\int f d\lambda = \sum_{i=1}^{n} \alpha_i \lambda(E_i).$$

It is easily established that this is a well-defined (positive) linear functional on $\mathcal{S}(\Sigma)$. Moreover,

$$\|f\|_1 = \int |f| d_\mu$$

is a pseudo-metric for $\mathcal{S}(\Sigma)$ with respect to which the integral is continuous. In fact, $|\int f d_\mu| \leq \|f\|_1$.

We can abstractly complete this space in the usual way: the completion consists of all equivalence classes $[(f_n)_1^\infty]$ of $\|\cdot\|_1$- Cauchy sequences $(f_n)_1^\infty$ in $\mathcal{S}(\Sigma)$, where $(f_n)$ is equivalent to $(g_n)$ provided $\|f_n - g_n\|_1 \to 0$. The norm of $[(f_n)]$ is given by $\|[(f_n)]\|_1 = \lim_{n\to\infty} \|f_n\|_1$.

The integral extends to the completion by

$$\int [(f_n)] d\lambda = \lim_{n\to\infty} \int f_n d\lambda$$

The integral is a continuous linear functional on the completion.

We shall identify this completion with a certain linear subspace of $\mathfrak{m}(\Sigma) / \mathfrak{n}(\lambda)$.

Suppose $\{f_n\}$ is a $\|\cdot\|_1$-Cauchy sequence in $\mathcal{S}(\Sigma)$. Then it is Cauchy in $\lambda$-measure, so there exists $f$ in $\mathfrak{m}(\Sigma)$ such that $f_n \to f$ in $\lambda$-measure. If $\{g_n\}$ is equivalent to $\{f_n\}$ and $g \in \mathfrak{m}(\Sigma)$ is such that $g_n \to g$ in measure, then since $f_n - g_n \to 0$ in measure, $f = g$ a.e.$[\lambda]$. Hence to each member $[\{f_n\}]$ of the abstract completion of $\mathcal{S}(\Sigma)$ there corresponds a unique $f + \mathfrak{n}(\lambda)$ in $\mathfrak{m}(\Sigma)/\mathfrak{n}(\lambda)$ such that $f_n \to f$ in measure. In fact, by 2.1.8, we may choose the representative $\{f_n\}$ of $[\{f_n\}]$ such that $f_n \to f$ a.e. The map $\varphi : [\{f_n\}] \to f + \mathfrak{n}(\lambda)$ is well-defined (and clearly linear) on the abstract completion of $(\mathcal{S}(\Sigma), \|\cdot\|_1)$ into $\mathfrak{m}(\Sigma)/\mathfrak{n}(\lambda)$.

We next show that $\varphi$ is one-to-one. Suppose $\{f_n\}_1^\infty$ is a Cauchy sequence in $\mathcal{S}(\Sigma)$ and suppose $f_n \to 0$ in measure. Since $\varphi$ is linear, we need only show $\|f_n\|_1 \to 0$. Given $\epsilon > 0$, choose $N$ such that $m,n \geqq N \Rightarrow \int |f_n - f_m| d\lambda < \epsilon/3$. Now fix $n \geqq N$. We will show $\int |f_n| d\lambda < \epsilon$ which will establish the result. Note that for any set $A$,

$$\int |f_n| \chi_A d\lambda \leqq \|f_n\|_\infty \lambda(A),$$

so there exists $\delta > 0$ such that

$$\lambda(A) < \delta \rightarrow \int |f_n| \chi_A d\lambda < \epsilon/_3.$$

Since $f_m \rightarrow 0$ in measure, there exists $m \geqq N$ such that

$$\lambda\{\omega: \ |f_m(\omega)| \geqq \epsilon/_3 \lambda(\Omega)\} < \delta.$$

Thus if $A = \{\omega : |f_m(\omega)| \geqq \epsilon/3\ \lambda(\Omega)\}$, then

$$\int |f_n| d\lambda = \int |f_n| \chi_A d\lambda + \int |f_n| \chi_{\Omega \setminus A} d\lambda < \frac{\epsilon}{3} + \int |f_n - f_m| \chi_{\Omega \setminus A} d\lambda$$

$$+ \int |f_m| \chi_{\Omega \setminus A} d\lambda$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3\lambda(\Omega)} \lambda(\Omega \setminus A) \leqq \epsilon .$$

Hence $\varphi$ is one-to-one.

2.3.2. DEFINITION. A function $f$ in $M(\Sigma)$ is $\lambda$-<u>integrable</u> if and only if there exists $(f_n)_1^\infty \subset \mathscr{S}(\Sigma)$ such that

(i) $f_n \rightarrow f$ a.e.$[\lambda]$,

and

(ii) $(f_n)_1^\infty$ is $\|\cdot\|_1$ - Cauchy.

In this case the $\lambda$-<u>integral</u> of $f$ is (uniquely) defined by

$$\int f d\lambda = \lim \int f_n d\lambda.$$

Thus $f \in \mathfrak{m}(\Sigma)$ is $\lambda$-integrable if and only if $f + \mathfrak{n}(\lambda)$ is in the range of $\varphi$. We let

$$L^1(\lambda) = \{f + \mathfrak{n}(\lambda) \quad : \quad f \text{ is } \lambda\text{-integrable}\}.$$

As is customary, we identify (or confuse) functions $f$ with equivalence classes $f + \mathfrak{n}(\lambda)$.

Note that if $\{f_n\}$ is a $\|\cdot\|_1$-Cauchy sequence in $\mathscr{S}(\Sigma)$, then so is $\{|f_n|\}$ (since $||f_n| - |f_m|| \leqq |f_n - f_m|$). If $f_n \to f$ a.e., then $|f_n| \to |f|$ a.e. The extension of the $\|\cdot\|_1$ - norm from $\mathscr{S}(\Sigma)$ to its completion is given by

$$\|[\{f_n\}]\|_1 = \lim_n \|f_n\|_1 = \lim_n \int |f_n| d\lambda = \int |f| d\lambda.$$

Thus when the norm is carried over to $L^1(\lambda)$, we have $\|f\|_1 = \int |f| d\lambda$ for all $\lambda$-integrable $f$. Hence, by construction, $L^1(\lambda)$ is a Banach space under the norm $\|f\|_1 = \int |f| d\lambda$.

If $f$ is $\lambda$-integrable and $f \geqq 0$, we can choose $f_n$ in $\mathscr{S}(\Sigma)$ such that $f_n \to f$ a.e. and $\{f_n\}$ is $\|\cdot\|_1$ - Cauchy. Then $|f_n| \to |f| = f$ a.e. and $\{|f_n|\}$ is $\|\cdot\|_1$ - Cauchy. Hence $\int f d\lambda = \lim \int |f_n| d\lambda \geqq 0$, and so the integral is a positive linear functional on $L^1(\lambda)$.

Also, if $f$ is in $\mathfrak{m}(\lambda)$ and $f \geq 0$, then $f$ is $\lambda$-integrable if and only if

$$s = \sup\{\int g d\lambda \ : \ 0 \leq g \leq f \ \text{a.e.}, \ g \in \mathcal{S}(\Sigma)\}$$

if finite. (If $s$ is finite, choose $(g_n) \subset \mathcal{S}(\Sigma)$ such that $0 \leq g_n \nearrow f$ a.e. Then $\int g_n d\lambda \nearrow$ some limit, and hence $(g_n)$ is $\|\cdot\|_1$ - Cauchy.) Thus if $0 \leq f \leq h$ and $h$ is integrable, then $f$ is integrable. It follows that for $f$ measurable, $f$ is integrable if and only if $|f|$ is integrable. For notational purposes, if $f$ is in $\mathfrak{m}(\Sigma)$ and $f \geq 0$, we let

$$\int f d\lambda = \sup\{\int g d\lambda : 0 \leq g \leq f \ \text{a.e.}, \ g \in \mathcal{S}(\Sigma)\} \leq \infty.$$

2.3.3. A sequence $(f_n)$ in $L^1(\lambda)$ is said to <u>converge in mean</u> to $f \in L^1(\lambda)$ (respectively, is <u>Cauchy in mean</u>) provided $\|f_n - f\|_1 \to 0$ ($\|f_n - f_m\|_1 \to 0$ as $m, n \to \infty$.)

If $f_n \to f$ in mean, then $\int f_n d\lambda \to \int f d\lambda$ (but not conversely) and $f_n \to f$ in measure (but not conversely).

2.3.4. THEOREM. (The Monotone Convergence Theorem).

<u>If $(f_n)$ is a sequence in $L^1(\lambda)$ and if</u>

(i) $f_{n+1} \geq f_n$ a.e., $\forall n$,

<u>and</u>

(ii) $\{\int f_n d\lambda\}$ <u>is bounded,</u>

<u>then $f_n$ converges in mean and a.e. to some member of $L^1(\lambda)$.</u>

PROOF. $\{\int f_n d\lambda\}$ is monotone increasing by (i) and hence has a limit by (ii). It follows that $\{f_n\}$ is Cauchy in mean.

2.3.5. COROLLARY. *If* $\{f_n\}$ *is a sequence in* $\mathfrak{m}(\Sigma)$, $f_n \geqq 0$ *a.e., and* $f_{n+1} \geqq f_n$ *a.e. for all* $n$, *then*

$$\int(\lim_{n\to\infty} f_n)\, d\lambda = \lim_n \int f_n d\lambda$$

PROOF. Either $\{\int f_n d\lambda\}$ is bounded or not.

2.3.6. COROLLARY (Fatou's Lemma). *If* $\{f_n\}$ *is a sequence in* $\mathfrak{m}(\Sigma)$ *and* $f_n \geqq 0$ *a.e. for all* $n$, *then*

$$\int(\lim\inf f_n)\, d\lambda \leqq \lim\inf\left(\int f_n d\lambda\right).$$

PROOF.
$$\begin{aligned}\int(\lim\inf f_n)d\lambda &= \int[\lim_m (\inf_{n\geqq m} f_n)]\, d\lambda\\ &= \lim_m \int(\inf_{n\geqq m} f_n)\, d\lambda\\ &\leqq \lim_m \inf_{n\geqq m}\left(\int f_n d\lambda\right)\\ &= \lim\inf\left(\int f_n d\lambda\right).\end{aligned}$$

2.3.7. THEOREM (Lebesque's Dominated Convergence Theorem).

<u>If</u> $\{f_n\}$ <u>is a sequence in</u> $L^1(\lambda)$, <u>if</u> $g$ <u>is in</u> $L^1(\lambda)$, <u>and if</u>

(i) $f_n \to f$ <u>a.e.</u>

<u>and</u>

(ii) $|f_n| \leqq g$ <u>a.e., all</u> $n$,

<u>then</u> $f$ <u>is in</u> $L^1(\lambda)$, <u>and</u>

$$\int f d\lambda = \lim \int f_n d\lambda .$$

PROOF. Since $|f| = \lim |f_n| \leqq g$ a.e., $f$ is in $L^1(\lambda)$.

Next, $0 \leqq g + f_n$ and $g + f_n \to g + f$. By Fatou's lemma,

$$\begin{aligned}\int f d\lambda &= \int (g + f) d\lambda - \int g d\lambda \\ &\leqq \liminf \left(\int (g + f_n) d\lambda\right) - \int g d\lambda \\ &= \liminf \left(\int f_n d\lambda\right).\end{aligned}$$

By symmetry,

$$\int (-f) d\lambda \leqq \liminf \int (-f_n) d\lambda = -\limsup \left(\int f_n d\lambda\right),$$

so that

$$\limsup \int f_n d\lambda \leqq \int f d\lambda \leqq \liminf \int f_n d\lambda.$$

2.3.8. NOTATION. If $f$ is $\lambda$-integrable (or measurable and non-negative), and if $E$ is in $\Sigma$, then

$$\int_E f d\lambda = \int f \chi_E d\lambda .$$

2.3.9. COROLLARY (Absolute Continuity of the Integral). If $(E_n)$ is a sequence in $\Sigma$, and if $\lambda(E_n) \to 0$, and if $f$ is in $L^1(\lambda)$, then

$$\lim_{n \to \infty} \int_{E_n} f d\lambda = 0$$

PROOF. $|f| \geq |\chi_{E_n} f| \to 0$ in measure. Every subsequence must have a sub-subsequence converging a.e. to 0, and the integrals converge to 0 by 2.3.7. Thus we must have

$$\int_{E_n} f d\lambda = \int f \chi_{E_n} d\lambda \to 0.$$

2.3.10. EXERCISES. (1) If $f$ is in $L^1(\lambda)$, then

$$\lim_{\lambda(E) \to 0} \int_E f d\lambda = 0$$

in the sense that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\lambda(E) < \delta$ implies $\left|\int_E f d\lambda\right| < \epsilon$ .

(2) If $f$ is in $L^1(\lambda)$, then $E \to \int_E f d\lambda$ is continuous on $(\Sigma, d)$ to $\mathbb{R}$.

(3) Use Egoroff's theorem and 2.1.8 to prove that if $g$ is in $L^1(\lambda)$, then in the set

$$\{f \in L^1(\lambda) \quad : \quad |f| \leq g\},$$

convergence in mean and convergence in measure coincide. Hence Lebesque's Dominated Convergence Theorem.

## 2.4 MEASURES ON COMPACT HAUSDORFF SPACES.

Throughout this section let $\Omega$ denote a compact Hausdorff space ($CT_2$-space).

2.4.1. The class of Baire subsets of $\Omega$ is the $\sigma$-algebra $B(\Omega)$ generated by the compact $G_\delta$ sets in $\Omega$.

Every continuous function $f:\Omega \to R$ is $B(\Omega)$-measurable since $f^{-1}(I)$ is a compact $G_\delta$ for every closed interval I.

2.3.2. Suppose $K$ is any compact subset of $\Omega$ and $U$ is an open set with $K \subset U$. By Urysohn's lemma there exists a continuous function $f:\Omega \to [0,1]$ such that $f(k) = \{1\}$ and $f(\Omega \setminus U) = \{0\}$. Thus there exists a compact $G_\delta$

$$K_0 = \{\omega ; f(\omega) = 1\}$$

and an open Baire set

$$U_0 = \{\omega : f(\omega) > \tfrac{1}{2}\}$$

such that

$$K \subset K_0 \subset U_0 \subset \bar{U}_0 \subset U.$$

In particular, $B(\Omega)$ contains a base for the topology of $\Omega$.

2.4.3. THEOREM. If $\Sigma$ is any $\sigma$-algebra in $\Omega$ that contains a base for the topology of $\Omega$, then $B(\Omega) \subset \Sigma$.

PROOF. Let $K$ be a compact $G_\delta$; say $K = \bigcap_n U_n$, where $U_n$ is open. Since $\Sigma$ contains a base for the topology, and since $K$ is compact, $\exists V_n \in \Sigma$ such that $K \subset V_n \subset U_n$ for each $n$. Hence $K = \bigcap_{n=1}^{\infty} V_n$ is in $\Sigma$.

2.4.4. THEOREM. Every compact set in $B(\Omega)$ is a $G_\delta$.

PROOF. Let $C$ be a compact set in $B(\Omega)$. Then $\exists (C_n)_1^\infty$, a sequence of compact $G_\delta$'s such that

$$C \in \Sigma((C_n))$$

(by 1.1.3). By Urysohn's lemma, for each $n$ $\exists$ a continuous function $f_n : \Omega \to [0,1]$ such that

$$C_n = \{\omega : f_n(\omega) = 0\}.$$

Define a pseudo-metric $d$ on $\Omega$ by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{|f_n(x)-f_n(y)|}{2^n} .$$

Note that the identity map $i:\Omega \to \Omega$ is continuous from the given topology to the d-topology on $\Omega$. Hence $C$ is d-compact.

Now let $\pi : \Omega \to \Omega/d$ be the natural quotient map, where $\Omega/d$ is the space obtained from $\Omega$ by identifying $x$ and $y$ if $d(x,y) = 0$. Note that $C_n = \pi^{-1}(\pi(C_n))$ since

$$\begin{aligned} x \in \pi^{-1}(\pi(C_n)) &\Rightarrow d(x,y) = 0 \text{ for some } y \text{ in } C_n \\ &\Rightarrow |f_n(x) - f_n(y)| = 0 \text{ some } y \text{ in } C_n \\ &\Rightarrow f_n(x) = 0 \\ &\Rightarrow x \in C_n. \end{aligned}$$

Let $\mathcal{S} = \{\pi^{-1}(A) : A \subset \Omega/d\}$. Then $\mathcal{S}$ is a $\sigma$-algebra containing $\{C_n\}$, and so $C = \pi^{-1}(A)$ for some set $A \subset \Omega/d$. Hence $\pi^{-1}(\pi(C)) = C$.

Now we show $C = \bigcap_{n=1}^{\infty} \{y : d(x,C) < \frac{1}{n}\}$, which will complete the proof. If $d(x,C) < \frac{1}{n}$, $\forall n$, choose $x_n \in C$ such that $d(x,x_n) \to 0$. Since $C$ is d-compact, we can assume $x_n \xrightarrow{d} x_0 \in C$. Then $d(x,x_0) = 0$, so $x \in \pi^{-1}(\pi(C)) = C$.

2.4.5. <u>For the remainder of this section, let $\lambda$ be a member of</u> $ca^+(B(\Omega))$.

2.4.6. THEOREM (Regularity of Baire measures). <u>For every set</u> $E$ <u>in</u> $B(\Omega)$

(i) $\lambda(E) = \inf\{\lambda(U)$: $U$ <u>is an open Baire set</u>, $E \subset U\}$. <u>and</u>

(ii) $\lambda(E) = \sup\{\lambda(K)$: $K$ <u>is a compact</u> $G_\delta$, $K \subset E\}$.

PROOF. Let $\mathcal{R}$ denote the collection of all sets $E \in B(\Omega)$ satisfying (i), and let $\mathcal{K}$ denote the collection of all compact $G_\delta$'s. Clearly $\mathcal{K} \subset \mathcal{R}$. We note the following facts:

(1) If $\{E_n\}_1^\infty \subset \mathcal{R}$, then $\bigcup_{n=1}^\infty E_n \in \mathcal{R}$. [For each $n$ choose an open Baire set $U_n$ with $E_n \subset U_n$ and $\lambda(U_n \setminus E_n) < \frac{\epsilon}{2^n}$. Then $\lambda[(\cup U_n) \setminus (\cup E_n)] \leq \lambda[\cup(U_n \setminus E_n)] < \epsilon$.]

(2) If $\{E_n\}_1^\infty \subset \mathcal{R}$ and $E_n \searrow E$, then $E \in \mathcal{R}$. [Choose $n$ such that $\lambda(E_n) < \lambda(E) + \epsilon/2$ and then choose $U \supset E_n \supset E$ such that $\lambda(U) < \lambda(E_n) + \epsilon/2 < \lambda(E) + \epsilon$.]

(3) If $C, D$ are in $\mathcal{K}$, then $C \setminus D$ is in $\mathcal{R}$. [Choose $U$ open with $C \subset U$ and $\lambda(U \setminus C) < \epsilon$. Then $U \setminus (C \cap D)$ is open, $(C \setminus D) \subset U \setminus (C \cap D)$, and $\lambda((U \setminus (C \cap D)) \setminus (C \setminus D)) = \lambda(U \setminus C) < \epsilon$.]

Since $\mathcal{K}$ is closed under finite unions and intersections, (3) and (1) together with 1.1.2 and 1.1.3 shows that the algebra generated by $\mathcal{K}$ is contained in $\mathcal{R}$.

But (1) and (2) imply $\mathcal{R}$ is a monotone class, so by 1.1.5, $B(\Omega) \subset \mathcal{R}$. This proves (i), and (ii) follows from (i) by complementation.

2.4.7. Let $C(\Omega)$ denote the linear space of all continuous functions $f: \Omega \to \mathbb{R}$. Since every such $f$ is $B(\Omega)$-measurable and bounded, it is integrable. The function $\varphi : C(\Omega) \to \mathbb{R}$ given by $\varphi(f) = \int f d\lambda$ is then a well-defined positive linear functional on $C(\Omega)$.

2.4.8. THEOREM. (The Riesz Representation Theorem). *If $\varphi$ is a positive linear functional on $C(\Omega)$, then there exists a unique Baire measure $\lambda \in ca^+(B(\Omega))$ such that*

$$\varphi(f) = \int f d\lambda$$

*for every $f$ in $C(\Omega)$.*

PROOF. The proof is omitted - see notes and references section.

2.4.9. A $CT_2$ - space $\Omega$ is *totally disconnected* if and only if for every open set $U$ and every point $x$ in $U$ there exists a set $V$ which is both open and closed (i.e., *clopen*) such that $x \in V \subset U$. That is, the clopen sets form a base for the topology of $\Omega$.

If $\Omega$ is totally disconnected, the collection $G$ of all clopen sets forms an algebra. Clearly $G \subset B^{*}(\Omega)$ since every clopen set is a compact $G_\delta$. On the other hand, $G$ contains a base for the topology of $\Omega$, and thus by 2.4.3,

$$B(\Omega) = \Sigma(G).$$

2.4.10. PROPOSITION. Let $\Omega$ be totally disconnected and let $G$ be the algebra of clopen sets. If $\{E_n\}_1^\infty$ is a disjoint sequence in $G$ and if $\bigcup_{n=1}^{\infty} E_n$ is in $G$, then all but finitely many $E_n$'s are empty.

PROOF. The non-empty members of $\{E_n\}_1^\infty$ form an open cover of $\bigcup_{n=1}^{\infty} E_n$ which has no finite subcover.

2.4.11. COROLLARY. If $\Omega$ is totally disconnected and $G$ is the algebra of clopen sets, then every finitely additive measure on $G$ is countably additive.

2.4.12. COROLLARY. If $\Omega$ is totally disconnected and $G$ is the algebra of clopen sets, then every member of $ba^+(G)$ has a unique extension to a member of $ca^+(B(\Omega))$.

R.2.1. REMARKS AND REFERENCES

(1) Frechet [1919] first introduced and studied the pseudo-metric $\rho$ on $M(\Sigma)$ as considered here.

(2) Note that throughout this chapter, $\lambda$ is bounded. If it is allowed to take the value $+\infty$, one must be slightly more careful. For example, the function $n$ in 2.1.1 can be defined in this case by

$$n(f) = \inf_{\epsilon>0} (\arctan [\epsilon + \lambda \{w: |f(w)| \geq \epsilon\}]),$$

and it is then real-valued and keeps the same properties (see e.g., Dunford and Schwartz [1958, p. 101 ff].

If $\lambda$ is allowed the value $+\infty$, Egoroff's theorem (2.1.6) fails, and so does 2.1.7. However, 2.1.8 and 2.1.9 remain valid in this setting (see Halmos [1950, §22]).

(3) Egoroff's theorem (2.1.6) is generalized to strongly measurable X-valued functions later (6.2.3), and so are 2.1.7, 2.1.8, and 2.1.9 (see 6.2.22, 6.2.23, and 6.2.24).

R.2.2. REMARKS AND REFERENCES

(1) Theorems 2.2.6 and 2.2.7 are special cases of Liapounov's theorem which states that the range of a finite dimensional non-atomic measure is closed and convex (see Liapounov [1940]).

(2) Theorem 2.2.10 was proved by Halmos and von Neumann [1942].

(3) In regard to Theorem 2.2.12, it is in fact true that $\Sigma/\lambda$ is compact if and only if $\lambda$ is purely atomic. (A quick proof can be given by considering the measure $\nu:\Sigma \to L^1(\lambda)$ given by $\nu(E) = \chi_E$ and using 7.1.4, 7.3.1, and 8.1.4 below. More elementary proofs can of course be found.)

Additional References: Hoffmann-Jørgensen [1971]

R.2.3. REMARKS AND REFERENCES

This is one of many approaches to the abstract Lebesque integral. The results here are standard material from courses in Real Variables; the theory of course generalizes to the case of the unbounded $\lambda$, see e.g. Halmos [1950].

R.2.4. REMARKS AND REFERENCES

(1) There is a vast theory concerning regular measures on compact and locally compact spaces. (For a beginning, see Halmos [1950].) We chose here only some results of immediate interest.

(2) A complete, very direct proof of the Riesz Representation Theorem (2.4.8) can be found in Rudin [1966].

(3) The material on totally disconnected spaces (2.4.9-2.4.12) will prove useful in §4.3ff.

Additional References: Dinculeanu and Kluvanek [1967]