VECTOR MEASURES

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1. INTRODUCTION

1.1. Classes of Sets

1.1.1. Recall that $\mathcal{A}$ ($\neq \phi$) is an <u>algebra</u> of subsets of a set Ω ($\neq \phi$) provided

$$\text{(i)} \quad A \in \mathcal{A} \Rightarrow \Omega \setminus A \in \mathcal{A}$$

and

$$\text{(ii)} \quad A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}.$$

An algebra is closed under finite unions and intersections and it contains ϕ and Ω. An algebra $\mathcal{A}$ is a σ-<u>algebra</u> provided

$$\text{(iii)} \quad \{A_n\}_1^\infty \subset \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

A σ-algebra is closed under countable intersections.

In general we will denote an algebra of sets by $\mathcal{A}$ and a σ-algebra of sets by Σ. If $\mathcal{E}$ is any class of sets, let $\mathcal{A}(\mathcal{E})$ and $\Sigma(\mathcal{E})$ denote the algebra and σ-algebra generated by $\mathcal{E}$, respectively.

1.1.2. If $\mathcal{E} = \{E_1, \ldots, E_n\}$ is a finite collection of subsets of Ω, the <u>atoms</u> of $\mathcal{E}$ are the sets of the form

$$A_J = \Big(\bigcap_{i \in J} E_i\Big) \setminus \Big(\bigcup_{j \notin J} E_j\Big),$$

where J ranges over the subsets of $\{1, \ldots, n\}$ (If $J = \phi$, $A_J = \Omega \setminus \big(\bigcup_{i=1}^{n} E_i\big)$, and if $J = \{1, \ldots, n\}$, $A_J = \bigcap_{i=1}^{n} E_i$.) The atoms form a finite partition of Ω and every member of $\mathcal{E}$ is a union of atoms. The collection of all unions of atoms is the algebra generated by $\mathcal{E}$.

1.1.3. If $\mathcal{E}$ is any collection of subsets of Ω, then

$$\mathcal{A}(\mathcal{E}) = \cup\{\mathcal{A}(\mathcal{F}) : \mathcal{F} \subset \mathcal{E},\ \ \mathcal{F}\ldots\text{finite}\}.$$

Indeed, the right hand side is an algebra which contains $\mathcal{E}$ and is contained in $\mathcal{A}(\mathcal{E})$. Similarly,

$$\Sigma(\mathcal{E}) = \cup\{\Sigma(\mathcal{C}) : \mathcal{C} \subset \mathcal{E},\ \ \mathcal{C}\ \text{countable}\}.$$

1.1.4. A collection $\mathcal{M}$ of subsets of Ω is called a <u>monotone class</u> if it is closed under limits of monotone sequences, i.e., if $\{A_n\}_1^\infty$ is a sequence in $\mathcal{M}$ with $A_{n+1} \subset A_n$ for all n (or $A_{n+1} \supset A_n$ for all n) then $\bigcap_{n=1}^\infty A_n$ (respectively, $\bigcup_{n=1}^\infty A_n$) is in $\mathcal{M}$.

A σ-algebra is clearly a monotone class. The next proposition proves very useful at times.

1.1.5. PROPOSITION. <u>If $\mathcal{A}$ is an algebra of sets, then $\Sigma(\mathcal{A})$ is the smallest monotone class containing $\mathcal{A}$.</u>

PROOF. Let $\mathcal{M}$ denote the smallest monotone class containing $\mathcal{A}$, i.e., $\mathcal{M}$ is the intersection of all monotone classes containing $\mathcal{A}$. We need only show that $\mathcal{M}$ is a σ-algebra since clearly $\mathcal{M} \subset \Sigma(\mathcal{A})$.

If $F \subset \Omega$, let

$$\mathcal{M}(F) = \{E \subset \Omega : E \setminus F \in \mathcal{M},\ F \setminus E \in \mathcal{M}, \text{ and } E \cup F \in \mathcal{M}\}.$$

Note that $\mathcal{M}(F)$ is a monotone class and that

$$E \in \mathcal{M}(F) \Rightarrow F \in \mathcal{M}(E).$$

We have

$$F \in \mathcal{A} \Rightarrow \mathcal{A} \subset \mathcal{M}(F) \Rightarrow \mathcal{M} \subset \mathcal{M}(F),$$

so

$$F \in \mathcal{A},\ E \in \mathcal{M} \Rightarrow E \in \mathcal{M}(F) \Rightarrow F \in \mathcal{M}(E),$$

and finally this gives

$$E \in \mathcal{M} \Rightarrow \mathcal{A} \subset \mathcal{M}(E) \Rightarrow \mathcal{M} \subset \mathcal{M}(E).$$

Thus $\mathcal{M}$ is closed under relative complements and finite unions, so $\mathcal{M}$ is both an algebra and a monotone class. Hence $\mathcal{M}$ is a σ-algebra.$\|$

1.1.6. Let $\mathcal{A}$ be an algebra of subsets of Ω, and let E be any subset of Ω. Let $\mathcal{P}_{\mathcal{A}}(E)$ (respectively, $\mathcal{P}^0_{\mathcal{A}}(E)$) denotes the collection of all countable (resp. finite) disjoint sequences $\pi = \{E_n\}$ in $\mathcal{A}$ for which $E \subset \bigcup_n E_n$. For $\pi_1 = \{E_n\}$ and $\pi_2 = \{F_m\}$ in $\mathcal{P}_{\mathcal{A}}(E)(\mathcal{P}^0_{\mathcal{A}}(E))$, let

$$\pi_1 \vee \pi_2 = \{E_n \cap E_m\}.$$

Then $\pi_1 \vee \pi_2$ is in $\mathcal{P}_{\mathcal{A}}(E)$ $(\mathcal{P}^0_{\mathcal{A}}(E))$. Write $\pi_1 \geq \pi_2$ if and only if $\pi_1 \vee \pi_2 = \pi_1$. Then $\mathcal{P}_{\mathcal{A}}(E)$ $(\mathcal{P}^0_{\mathcal{A}}(E))$ is a directed set (in fact, it is an upper semi-lattice.) We shall often consider nets whose domain are $\mathcal{P}_{\mathcal{A}}(E)$ $(\mathcal{P}^0_{\mathcal{A}}(E))$.

Note that in the case $E = \Omega$, $\mathcal{P}_{\mathcal{A}}(E)$ $(\mathcal{P}^0_{\mathcal{A}}(E))$ is simply the collection of all countable (finite) $\mathcal{A}$-partitions of Ω. We write $\mathcal{P}_{\mathcal{A}}(\mathcal{P}^0_{\mathcal{A}})$ for $\mathcal{P}_{\mathcal{A}}(\Omega)$ $(\mathcal{P}^0_{\mathcal{A}}(\Omega))$.

1.2. SET FUNCTIONS (= MEASURES)

1.2.1. Given an algebra A of sets, and a Banach space X, a function $\mu : A \to X$ is said to be finitely additive (f.a.) provided

$$A, B \in A,\ A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B).$$

Moreover, μ is said to be countably additive (c.a.) provided for every disjoint sequence $\{A_n\}_1^\infty$ in A,

$$\bigcup_{n=1}^{\infty} A_n \in A \Rightarrow \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n),$$

where the last infinite sum means $\lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n)$, the limit being taken in the norm topology of X.

We shall occasionally consider non-negative real-valued measures which may also take the value $+\infty$, e.g., Lebesgue measure on $\mathbb{R}$ and counting measure on arbitrary sets.

1.2.2. We let $ba(A,X)$ denote the Banach space of all bounded f.a. measures $\mu : A \to X$ with the uniform norm,

$$\|\mu\|_\infty = \sup\{\|\mu(A)\| : A \in A\}.$$

The closed subspace of $ba(A,X)$ consisting of all countably additive members of $ba(A,X)$ will be denoted by $ca(A,X)$.

For ease of notation we write $ba(A)$ and $ca(A)$ for $ba(A,\mathbb{R})$ and $ca(A,\mathbb{R})$, respectively. Also, $ba^+(A)$ and $ca^+(A)$ denotes the sets of non-negative members of $ba(A)$ and $ca(A)$.

1.2.3. Remarks about boundedness. (1) A non-negative, real-valued f.a. measure on A is of course bounded automatically. However, there exist real-valued f.a. (even c.a.) measures on algebras which are not bounded. [Let

A denote the algebra of all finite and co-finite subsets of R. Define μ on A by $\mu(\emptyset) = 0$,

$$\mu(\{a_1,\ldots,a_n\}) = n, \quad \text{and} \quad \mu(R\setminus\{a_1,\ldots,a_n\}) = -n.$$

Then μ is real-valued and c.a., but not bounded.)

(2) The Hahn-Jordan decomposition theorem (see §3.1 below) implies that any real-valued countably additive measure on a σ-algebra is bounded. If $\mu:\Sigma \to X$ is c.a., then for each x^* in X^*, $x^* \circ \mu:\Sigma \to R$ is c.a., hence bounded. It then follows from the uniform boundedness principle (§A.2) that μ is bounded. Thus any c.a. measure on a σ-algebra to a Banach space is automatically bounded. However, there exist f.a. measures $\mu:\Sigma \to X$ (even with $X = R$) which are not bounded (All one needs is the existence of a non-continuous linear functional on the Σ-simple functions, see 4.1.2 below.)

1.2.4. THEOREM (Extension Theorem). <u>If μ is a non-negative member of</u> $ca(A)$, <u>then μ has a unique extension to a c.a. measure $\overline{\mu}$ on $\Sigma(A)$ given by</u>

$$(\#) \qquad \overline{\mu}(E) = \inf \sum_{n=1}^{\infty} \mu(E_n) \qquad (E \in \Sigma(A)),$$

<u>where the infimum is taken over all disjoint sequences $\{E_n\}$ in A with</u> $E \subset \bigcup_{n=1}^{\infty} E_n$.

PROOF. (Outline). We assume this result is familiar to the reader and only outline the proof. For complete details, see any standard text.

Define $\overline{\mu}$ on all subsets E of Ω by formula (ℓ). Then $\overline{\mu}$ is a Caratheodory outer measure on 2^Ω, i.e.,

(i) $\overline{\mu}(\emptyset) = 0$

(ii) $A \subset B \subset \Omega \Rightarrow 0 \le \overline{\mu}(A) \le \overline{\mu}(B) < \infty$,

and

(iii) $\overline{\mu}(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \overline{\mu}(E_n)$, $\forall \{E_n\}_1^\infty \subset 2^\Omega$.

The class of all $\overline{\mu}$-measurable sets (.e., all $E \subset \Omega$ which satisfy $\overline{\mu}(A) = \overline{\mu}(A \cap E) + \overline{\mu}(A \setminus E), \forall A \subset \Omega$) is a σ-algebra on which $\overline{\mu}$ is a c.a. measure. The members of A are $\overline{\mu}$-measurable, and so $\Sigma(A)$ consists of $\overline{\mu}$-measurable sets. Hence $\overline{\mu}$ restricted to $\Sigma(A)$ is c.a. and extends μ.

Uniqueness can be proved directly; it also follows from 2.2.3 below.‖

1.2.5. A <u>lattice</u> of subsets of Ω is a collection L which is closed under finite unions and intersections. If E is any collection of subsets of Ω, $L(E)$ denotes the smallest lattice containing E.

1.2.6. THEOREM (Extension Theorem). <u>Let</u> L <u>be a lattice of subsets of</u> Ω, <u>let</u> X <u>be a Banach space, and let</u> $\lambda : L \to X$ <u>be a function such that</u>

$$\lambda(A \cup B) = \lambda(A) + \lambda(B) - \lambda(A \cap B) \qquad (A, B \in L)$$

<u>and</u>

$$\lambda(\emptyset) = 0$$

if ϕ *is in* L. *If* Ω *is in* L, *let* $x_0 = \lambda(\Omega)$; *otherwise let* x_0 *be an arbitrary member of* X.

There exists a unique extension of λ *to a finitely additive measure* $\mu: A(L) \to X$ *such that* $\mu(\Omega) = x_0$.

PROOF. A standard induction argument shows that if $A_1, \ldots, A_n$ are in L, then

$$\lambda(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} (-1)^{i+1} S_i,$$

where for each $i = 1, \ldots, n$,

$$S_i = \sum\{\lambda(A_{j(1)} \cap \ldots \cap A_{j(i)}) : 1 \leq j(1) < \ldots < j(i) \leq n\}.$$

In particular,

(#) if two such functions λ agree on all the A_i's and on all their intersections, then they agree on their union.

We first prove the theorem if L is finite, say $L = \{L_1, \ldots, L_n\}$. Without loss of generality assume ϕ is in L and $\lambda(\phi) = 0$. For each atom (see 1.1.2)

$$A_J = (\bigcap_{i\in J} L_i)\setminus(\bigcup_{j\notin J} L_j)$$

of L, let

$$\mu(A_J) = \begin{cases} \lambda(\bigcap_{i\in J} L_i) - \lambda[(\bigcap_{j\notin J} L_j)\cap(\bigcup_{i\in J} L_i)] & \text{if } \emptyset \neq J \subsetneq \{1,\ldots,n\} \\ x_0 - \lambda(\bigcup_{i=1}^{n} L_i) & \text{if } J = \emptyset \\ \lambda(\bigcap_{i=1}^{n} L_i) = \lambda(\emptyset) = 0 & \text{if } J = \{1,\ldots,n\}, \end{cases}$$

and extend μ to be additive on $A(L)$ (= the set of all unions of the atoms.) We prove by (reverse) induction on $k = n, n-1, \ldots, 1$ that $J \subset \{1,\ldots,n\}$ and card $(J) = k$ implies

$$(*) \qquad \lambda(\bigcap_{i\in J} L_i) = \mu(\bigcap_{i\in J} L_i).$$

If $k = n$, then (*) holds by definition of μ. Now suppose (*) holds for card $(J) = k, k+1, \ldots, n$, where $1 < k \leq n$. Let K be a subset of $\{1,\ldots,n\}$ with card $(K) = k-1$. Note that

$$(\bigcap_{i\in K} L_i) = [(\bigcap_{i\in K} L_i)\setminus(\bigcup_{j\notin K} L_j)]\cup[(\bigcup_{i\notin K} L_j)\cap(\bigcap_{i\in K} L_i)]$$

$$= [(\bigcap_{i\in K} L_i)\setminus(\bigcup_{j\notin K} L_j)]\cup[\bigcup_{j\notin K} (L_j\cap(\bigcap_{i\in K} L_i))].$$

By the induction hypothesis and (#),

$$\mu[\bigcup_{j\notin K}(L_j \cap (\bigcap_{i\in K} L_i))] = \lambda[\bigcup_{j\notin K}(L_j \cap (\bigcap_{i\in K} L_i))].$$

Therefore, by definition of μ,

$$\lambda(\bigcap_{i\in K} L_i) = \mu[(\bigcap_{i\in K} L_i)\setminus(\bigcup_{j\notin K} L_j)] + \lambda[\bigcup_{j\notin K}(L_j \cap (\bigcap_{i\in K} L_i))]$$

$$= \mu[(\bigcap_{i\in K} L_i)\setminus(\bigcup_{j\notin K} L_j)] + \mu[\bigcup_{j\notin K}(L_j \cap (\bigcap_{i\in K} L_i))]$$

$$= \mu(\bigcap_{i\in K} L_i).$$

Letting J range over the singleton sets, (*) implies that μ extends λ. Uniqueness is clear from the definition, and this completes the proof of the theorem if L is finite.

For the general case, recall that (by 1.1.3)

$$A(L) = \bigcup\{A(E) : E \subset L,\ E \text{ finite}\}.$$

By the above, for each finite $E \subset L$ there is a unique f.a. μ_E on $A(E) = A(L(E))$ which extends $\lambda | L(E)$. If E and F are two finite subsets of L, then by uniqueness $\mu_{E\cup F} | A(E) = \mu_E$ and

1.2.6.

$\mu_{E\cup F}|A(F) = \mu_F$; hence μ_E agrees with μ_F on $A(E) \cap A(F)$. Therefore one can knit the μ_E's together to obtain a function $\mu: A(L) \to X$; clearly μ is finitely additive, extends λ, takes the value x_0 on Ω, and is unique with respect to these properties.

R.1.1. REMARKS AND REFERENCES

Most of the results presented in these notes have extensions to cases where the measures have more general domains than algebras or σ-algebras (also called (Boolean) fields and σ-fields) such as rings, δ-rings, σ-rings, etc. However, we shall stay with the basic cases in the Notes, and only occasionally point out generalizations in the Remarks. For a more complete discussion of various important types of classes of sets, see e.g. Halmos [1950, Chapter 1].

R.1.2. REMARKS AND REFERENCES

(1) As with domains of the measures (see R.1.1), many of the results presented in these notes have extensions to cases where the measures have more general range spaces than Banach spaces. For example, Theorem 1.2.6 and its proof hold if X is any additive group. However, in the Notes all our measures will take their values in Banach spaces.

(2) The usual (and equivalent) statement of Theorem 1.2.6 is that λ has a unique extension to a finitely additive function on the ring generated by L. Theorem 1.2.6 was first formally stated and proved by B.J. Pettis [1951, Th. 1.2] and it has been rediscovered and published several other times (see Pettis' review of a proof given by Lipecki (MATH REVIEWS, vol. 44, #7586, p. 1377 (1972))). The proof given here is due to Huff [1969, unpublished]; it is similar to that of Lipecki.

<u>Additional References:</u> Kisynski [1968], Huneycutt [1969].