

ABSTRACT. Let X be a normed linear space and $S(X) = \{x \in X : \|x\| = 1\}$ be its unit sphere. For a given $x \in S(X)$, let $E(X)$ denote the minimal extremal subset of $S(X)$ which contains x . In this paper, the minimal extremal subsets are studied both in general spaces and in some concrete ones. It turns out that much can be said about the general facial structure of $S(X)$ with just the knowledge of its minimal faces. (A face of $S(X)$ is convex extremal subset.) We will be particularly interested in studying those spaces X having either i) the Q -property, i.e. $E(x) = Q(x)$ for each $x \in S(X)$, where $Q(x)$ is the intersection with $S(X)$ of all the supporting hyperplanes to $S(X)$ at x , or ii) the weak Q -property, i.e. $\overline{E(x)} = Q(x)$ for each $x \in S(X)$. In particular (Theorem 3.3), every space of type $C_0(T)$, with T locally compact Hausdorff, always has the weak Q -property; while (Theorem 3.7) $C_0(T)$ has the Q -property if and only if T is discrete. Similarly (Theorem 4.2), a space of type $L_1(T, \Sigma, u)$, with (T, Σ, u) a σ -finite measure space, always has the weak Q -property; while (Theorem 4.7) $L_1(T, \Sigma, u)$ has the Q -property if and only if (T, Σ, u) is a finite union of atoms. Further (Theorem 3.4 and 4.3), in every space X of type $C_0(T)$ or $L_1(T, \Sigma, u)$, the closure of every face in $S(X)$ is again a face. As opposed to this result however, an example (Proposition 3.10) is constructed of a Banach space X and an element $x \in S(X)$ such that $\overline{E(x)}$ is not a face. In fact (Proposition 5.5), X may be chosen to be separable and reflexive. (To our knowledge, this appears to be the first example of a face of a unit sphere whose closure is not extremal.)