Homework 9, Due November 7th

1. For $(x,t) \in [0,\infty) \times \mathbb{R}$ consider the nonhomogeneous Burgers' equation

$$u_t + uu_x + \sin(x) = 0, \;\; u(x,0) = 0.$$

Find the largest time t, so that the solution could be determined on $[0,t)$ by the method of characteristics.
Note: it turns out that in this problem the main effort is in the analysis of the characteristic ODE (e.g. when do (projected) characteristics intersect for the first time?). Please, provide sufficient details of this analysis.

2-5. Problems 5-8 from Evans on p. 163-164

Extra problems (for your practice only, do not submit solutions). Here I give problems from the midterm exam.

6 1. On the unit disc $\Omega = \{x^2 + y^2 < 1\}$, consider the (elliptic) equation

$$\Delta u = \alpha e^x \qquad x \in \Omega$$

with boundary data

$$u(x, y) = x + y^3 \qquad (x, y) \in \partial\Omega\,.$$

Prove that if $\alpha = 0$ then at the origin one has $u(0, 0) = 0$. On the other hand, if the constant α is strictly positive, then $u(0, 0) < 0$.
Hint: Observe that the boundary data $x + y^3$ is an odd function.

2. On the disc $\Omega = \{x^2 + y^2 < 4\}$, consider the nonhomogeneous heat equation

$$u_t - \Delta u = 1,\ u(x, y, t) = 0 \text{ on } \partial\Omega,\ u(x, y, 0) = (x^2 + y^2)^2/16 - (x^2 + y^2)/4.$$

Prove that the solution is monotone increasing in time, i.e. $u_t > 0$ for all $t \geq 0$ and $(x, y) \in \Omega$.

3. a) Derive a formula similar to D'Alembert's formula for the solution $u(x, t)$ of the wave equation on the a unit interval $\Omega =]0, 1[$:

$$u_{tt} - u_{xx} = 0 \qquad 0 < x < 1, \quad t > 0,$$

with boundary conditions $u_x(0, t) = 0,\ u(1, t) = 0, \quad t \geq 0$, and initial conditions

$$u(x, 0) = f(x),\ u_t(x, 0) = g(x), \text{ for } 0 < x < 1,\ f(0) = f'(1) = g(0) = g'(1) = 0.$$

b) Prove that the total energy

$$E(t) = \frac{1}{2}\int_0^1 \left(u_t^2(x, t) + u_x^2(x, t)\right)\, dx,$$

is constant in time.

Note: In part a) you do not have to give the formula explicitly in terms of $f(x)$ and $g(x)$. It suffices to explain each term in your final expression.

4. For $(x, t) \in [0, \infty) \times \mathbb{R}$ consider the nonhomogeneous Burgers' equation

$$u_t + uu_x + x = 0,\ u(x, 0) = 0.$$

a) Do characteristics intersect? If yes, find the first time they intersect.
b) Find the solution of this equation.
Note: For part b) you need to find the global solution $u(x, t)$ for all times if characteristics do not intersect, or the solution $u(x, t)$ for $0 < t < t_0$ if characteristics intersect at t_0.