Homework 8, Due October 26th

1. Prove the local uniqueness theorem:
Let $F, g \in C^3$ and consider the problem

$$F(Du, u, x) = 0,\ u(x) = g(x) \text{ on } \Gamma. \tag{1}$$

Let $x_o \in \Gamma$ be a noncharacteristic point. Then there exists a constant δ depending on F, g and and $Du(x_o)$, such that if u_1 and u_2 are two (local) solutions of (1) near x_o with the (key) property $|Du_1(x_o) - Du_2(x_o)| < \delta$, then $u_1 \equiv u_2$.

2. Consider the inviscid Burgers' equation

$$u_t + uu_x = 0\,, \qquad u(x,0) = \phi(x)\,, \qquad x \in \mathbb{R}\,.$$

(a) Show that the function $v(x,t) := u_x(x,t)$ satisfies the equation

$$v_t + uv_x = -v^2.$$

(b) Define the curve $x(t)$ in the (x,t)-plane by $v(x(t),t) = \min_{x\in\mathbf{R}} v(x,t)$. Argue that $u_{xx}(x(t),t) = 0, \qquad$ for all t.
(c) Show that $V(t) := v(x(t),t)$ satisfies the ODE $\dot{V}(t) = -V(t)^2$.
(d) Use this to argue that: if there is a point x such that $\phi'(x) < 0$, then the gradient u_x of the solution of Burgers' equation reaches $-\infty$ in finite time.

3. Let $u \in C^1$ be a solution of a (linear) equation $a_1(x)u_1 + a_2(x)u_2 = -u$ on $B(0,1)$. Prove that $u \equiv 0$, if $a_1x_1 + a_2x_2 > 0$ on $\partial B(0,1)$.
Hint: show that $\max u \leq 0$ and $\min u \geq 0$.

4. Problem 4 from Evans. pp.163.

5. Solve

$$u = xu_x + yu_y + \frac{1}{2}(u_x^2 + u_y^2),\ u(x,0) = (1 - x^2)/2.$$

Extra problem (for your practice only, do not submit solutions).

6. Liouville equation

Consider the Hamilton's system

$$\dot{x} = D_p H, \;\; \dot{p} = -D_x H$$

Every solution of this system describes a trajectory of moving particle, where $x(t)$ is the position and $p(t)$ is the velocity. Imagine that we do not know the precise initial conditions of the particle at $t = 0$, but we only know its probability density ρ_0:

$$\int_A \rho_0(p, x) dx dp = \text{Prob}\{\text{Particle is initially at } x_0 \text{ with velocity } p_0, \; (x_0, p_0) \in A\}.$$

a) Let ρ_t be the probability density at time t. Find the equation for ρ_t. This is the Liouville equation.

b) Work-out the special case

$$H = \frac{p^2}{2m} + U(x),$$

where U is smooth.

Note: use the fact that the Hamiltonian is constant along characteristics. Also, the answer in the linear case is given at the end of p.3 of Evans.