Homework 6, Due October 10th

1-3. Problems 16-18 from Evans. p.89.

4. Consider the following problem for the linear wave equation in $\mathbb{R}^3$:

$$u_{tt} - c^2 \Delta u = 0 \tag{1}$$
$$u(t,0) = \varphi(t)\,. \tag{2}$$

Here $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, and φ is a given smooth function. Find a radially invariant solution of the above wave equation.
Note: u should be such that it is a function only of t and $r = |x|$. You will recover here what we discussed in class, when we talked about spherical waves. Note also that this is NOT an initial value problem - values at $x = 0$ and t arbitrary are prescribed.

5. a) Derive an explicit formula similar to D'Alembert's formula for the solution $u(t,x)$ of the wave equation on the half-line $\Omega =]0,\infty[$:

$$u_{tt} - c^2 u_{xx} = 0 \qquad x > 0, \quad t > 0,$$

with initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \qquad x > 0,$$

and boundary condition

$$u_x(0,t) = 0 \qquad t \geq 0\,.$$

b) Assume that f and g are smooth and with compact support. Is the total energy

$$E(t) = \frac{1}{2}\int_0^\infty u_t^2(x,t) + c^2 u_x^2(x,t)\,dx\,.$$

constant in time ? Prove it.

Extra problems (for your practice only, do not submit solutions).

6. Consider the following homogeneous Dirichlet problem on an open domain $\Omega \subset \mathbb{R}^n$ (with smooth boundary $\partial\Omega$):

$$
\begin{aligned}
u_t - \Delta u &= 0 \qquad \text{in } (0,T] \times \Omega && (3)\\
u(t,x) &= 0 \qquad \text{on } [0,T] \times \partial\Omega && (4)\\
u(0,x) &= u_0(x) \qquad \text{for } x \in \Omega\,. && (5)
\end{aligned}
$$

Show that

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \;\leq\; \frac{\|u_0\|_{L^1(\Omega)}}{(4\pi t)^{\frac{n}{2}}}$$

7. Hadamard's Three Circle Theorem.

Let $U \subset \mathbb{R}^2$ be open and connected, and $u \in C^0(\bar{U})$ is said to be *locally subharmonic* (LSH) on U if for every ball $B \subset U$, with $\bar{B} \subset U$, and for each harmonic function $h \in C^2(B) \cap C^0(\bar{B})$, we have

$$u \leq h \quad \text{on } \partial B \quad \Longrightarrow \quad u \leq h \quad \text{in } B.$$

One can show that any non-constant LHS function on U attains its maximum *only* at the boundary of U.

The above formulation is yet another way to look at subharmonic functions. It is instructive to work out how subharmonic functions in our sense are also subharmonic functions in the sense above.

Assume that the function $u(x,y)$ is LHS on an open set in $\mathbb{R}^2$ which contains the annulus $r_1^2 \leq x^2 + y^2 \leq r_2^2$. Prove that the maximum

$$M(r) = \max_{x^2+y^2=r^2} u(x,y)$$

satisfies

$$M(r) \leq \frac{\ln\left(\frac{r_2}{r}\right) M(r_1) + \ln\left(\frac{r}{r_1}\right) M(r_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

for $r_1 < r < r_2$.