Homework 5, Due October 3rd

1. Problem 10 from Evans. p.87.
2. Find *traveling wave* solutions of the heat equation

$$u_t = k\Delta u$$

in $\mathbb{R}^n$. That is determine all (smooth) solutions in the form

$$u(x,t) = f(b \cdot x - ct),\ |b| = 1.$$

3. Prove that there is at most one classical solution of the following equations considered on a bounded domain $\Omega \in \mathbb{R}^n$.

a) Advection-diffusion equation:

$$u_t + v \cdot \nabla u - \Delta u = f(x,t) \text{ on } \Omega \times [0,T],\ u = g \text{ at } t = 0,\ u = 0, \text{ in } \partial\Omega,$$

with a smooth incompressible flow v $\nabla \cdot v = 0$.

b) Heat equation with Neumann boundary conditions:

$$u_t - \Delta u = f(x,t) \text{ on } \Omega \times [0,T],\ u = g \text{ at } t = 0,\ \partial u/\partial \nu = 0 \text{ in } \partial\Omega.$$

4. Prove (weak) maximum principle for the advection-diffusion equation

$$u_t + v \cdot \nabla u - \Delta u = 0 \text{ on } \Omega \times [0,T],\ u = g \text{ at } t = 0,\ u = 0, \text{ in } \partial\Omega.$$

That is show that

$$\max_{\bar{\Omega}\times[0,T]} |u| \leq \max_{\bar{\Omega}} |g|.$$

Hint: Use energy methods. Observe that

$$d_t \int_\Omega |u|^p dx \leq 0$$

for *any* $p > 0$ (it suffices to assume that p is an even integer) and show that for (bounded) functions

$$||u||_{L^\infty} = \lim_{p\to\infty} ||u||_{L^p}.$$

5. Problem 15 from Evans. p.88.

Extra problems (for your practice only, do not submit solutions).

6. Prove Poincare inequality: suppose $u \in C^1(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$, then

$$||\nabla u||^2_{L^2(\Omega)} \leq C||u||^2_{L^2(\Omega)}$$

with a universal constant, that depends on the domain Ω only.

7. Prove exponential decay of energy: for the solution of

$$u_t - \Delta u = 0 \text{ on } \Omega \times [0, T],\ u = g \text{ at } t = 0,\ u = 0, \text{ in } \partial\Omega,$$

we have

$$||u(\cdot, t)||_{L^2(\Omega)} \leq e^{-Ct}||g||_{L^2(\Omega)}.$$

Hint: Use Poincare inequality.

8. Prove exponential decay of energy for the solution of an advection-diffusion equation

$$u_t + v \cdot \nabla u - \Delta u = 0 \text{ on } \Omega \times [0, T],\ u = g \text{ at } t = 0,\ u = 0, \text{ in } \partial\Omega,$$

with a smooth incompressible flow v $\nabla \cdot v = 0$. Namely, show that

$$||u(\cdot, t)||_{L^2(\Omega)} \leq e^{-Ct}||g||_{L^2(\Omega)},$$

where the constant C is independent of v