Homework 4, Due September 26th

1. Prove that any non-negative solution $u \in C^2(\Omega)$ of the stationary reaction-diffusion equation

$$-\Delta u = f(u) \text{ in } \Omega, \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega,$$

with KPP (Kolmogorov-Petrovsky-Piskunov) reaction rate

$$f(u) = u^p(1-u), \ p > 0,$$

is $u \equiv 0$ or $u \equiv 1$. Here Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary.

2. a) Show that there is no classical (i.e. $C^1([-1,1])$) solutions of the one-dimensoinal eikonal equation

$$(u')^2 = 1, \text{ on } [-1,1], \ u(-1) = u(1) = 0.$$

b) Consider a two-well potential:

$$I[w] = \int_{-1}^{1} (1 - (w'(x))^2)^2 dx, \ u(-1) = u(1) = 0.$$

Prove that $I[w]$ has no minimizers on $C^1([-1,1])$.
c) Construct several minimizers of $I[w]$ $C([-1,1])$, thus showing their non-uniqueness.
d) Construct explicitly a family of solutions $u_n \in C([-1,1])$ of the (weak) eikonal equation

$$(u_n')^2 = 1, \text{ almost everywhere on } [-1,1], \ u_n(-1) = u_n(1) = 0,$$

such that $u_n \to u$ uniformly (i.e $||u_n - u||_{C[-1,1]}$), and

$$(u')^2 \neq 1 \text{ on } [-1,1].$$

3. Problem 11 (Chapter 2) on p.87 of Evans's book.
4. Problem 13 (Chapter 2) on p.87 of Evans's book.
5. Problem 14 (Chapter 2) on p.88 of Evans's book.

Extra problems (for your practice only, do not submit solutions).

6. Suppose $u \in C^2(\Omega)$ is the solution of

$$-\Delta u = f(x), \text{ in } \Omega, u = 0, \text{ on } \partial\Omega,$$

with $f \geq 0$, $\int_\Omega f dx > 0$. Prove that for any $x \in \Omega$, $u(x) > 0$. Here Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary.

Note: This illustrates "non-locality" of the Laplacian: $f(x)$ may be zero almost everywhere, but it is positive on a set of small measure. This is enough to conclude that $u > 0$ everywhere inside the domain.

7. Viscous regularization. For small $\epsilon > 0$ consider a regularization of the eikonal equation

$$-\epsilon u'' + (u')^2 = 1, \text{ on } [-1, 1], \; u(-1) = u(1) = 0.$$

a) Show that this equation has a unique solution.
b) Show that there is a limit $\epsilon \to 0$ these solutions.
c) Show that the limiting function solves the (weak) eikonal equation

$$(u_n')^2 = 1, \text{ almost everywhere on } [-1, 1], \; u_n(-1) = u_n(1) = 0.$$

8. Suppose $\Phi(x - y)$ is the fundamental solution of the Poisson equation on an open bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. For $x \in \Omega$ let $\phi^x(y)$ be the corrector

$$\Delta\phi^x = 0, \text{ in } \Omega, \; \phi^x = \Phi(x - y) \text{ on } \partial\Omega.$$

a) Prove that

$$\phi^x(y)dy \leq \Phi(x - y)(y).$$

b) Prove that problem 5 p.86 from Evans is true not only for a ball, but for any bounded domain.

9. Hopf's Lemma. Suppose $u \in C^2(B(0, 1)) \cap C^1(\bar{B}(0, 1))$ is the solution of the Laplace equation on a unit ball:

$$\Delta u = 0, \text{ in } B(0, 1), \; u = f, \text{ on } \partial B(0, 1).$$

Let $x_0 \in \partial B(0,1)$, be the point where u achieves its maximum. Prove that the normal derivative at this point is *strictly* positive:

$$\frac{\partial u}{\partial \nu} > 0.$$

Note: $\frac{\partial u}{\partial \nu} \geq 0$ is obvious.

10. Suppose a sequence u_n of harmonic functions converges uniformly on compact sets to u. Prove that u is harmonic.

Note: it means, in particular that the space of harmonic functions on a bounded set Ω is a closed subset in $C^0(\Omega)$, i.e it is a Banach space.