Homework 11, Due December 5th

1. Total variation

Recall the definition of (total) variation given in part 6 of the proof of the uniqueness of entropy solutions theorem (p 153 in Evans):

$$var_x f(x,t) = \sup_{x_i, N} \sum_{i=1}^{N} |f(x_i,t) - f(x_{i+1},t)|.$$

a) Prove that for a linear homogeneous first order equation

$$u_t + b(x,t) \cdot \nabla u = 0,\ x \in \mathbb{R}^n,\ u(x, t=0) = g(x), b \in C_0^\infty$$

the total variation of the solution does not change: for any t

$$var_x u(x,t) = var_x g(x).$$

b) Prove that for a conservation law with uniformly convex smooth F:

$$u_t + F'(u) u_x = 0,\ x \in \mathbb{R},\ u(x,t=0) = g(x),\ g(x) \text{ is piece-wise contnuous}$$

the total variation of the entropy solution does not increase: for any $t \geq 0$ and $\tau \geq 0$

$$var_x u(x, t+\tau) \leq var_x u(x,t).$$

c) Solve

$$u_t + u u_x = 0,\ x \in \mathbb{R},\ u(x, t=0) = \begin{cases} 0,\ x < 0, \\ 1,\ 0 \leq x \leq 1, \\ 0,\ x > 1, \end{cases}$$

and verify that total variation may be *strictly* decreasing.

Note: for part c) the solution is given in example 3 on p.143 of Evans.

2-5. Problems 11-14 from Evans on p.164