Homework 10, Due November 14th

1. (Kruzhkov's method)

a) Show that if u is the solution of the viscous Hamilton-Jacobi equations

$$u_t + \frac{1}{2}|\nabla u|^2 = \frac{\varepsilon}{2}\Delta u, \ u = u(x,t), x \in \mathbb{R}^n.$$

Then $v^\varepsilon = e^{-u/\varepsilon}$ is the solution of the heat equation

$$v_t = \frac{\varepsilon^2}{2}\Delta v.$$

b) Consider the one-dimensional case. Let

$$u(x,0) = \begin{cases} 0, \ x > 0 \\ 2x, \ x < 0. \end{cases}$$

Find $v^\varepsilon(x,t)$ for each ε. Observe that as $\varepsilon \to 0$, then $v^\varepsilon \to v$. What is v? Note: It might be useful to look at the Hopf-Cole transformation in the textbook at p.194 and p.205.

2. (Riemann's problem)

Let $H(p)$ is a C^2 uniformly convex even function with $H(0) = 0$,

$$g(x) = \begin{cases} ax, \ x > 0 \\ bx, \ x < 0. \end{cases}$$

with some constants a and b. Find all the traveling wave solutions $u(x,t) = g(x - ct)$ of the one-dimensional Hamilton-Jacobi equation $u_t + H(u_x) = 0$. Hint: First figure out the relations between a and b when there is no traveling wave solutions of the desired form.

3. (Importance of Lipschitz initial conditions)
Solve, using Hopf-Lax formula the one-dimensional Hamilton-Jacobi equation

$$u_t + \frac{1}{2}u_x^2 = 0$$

where

$$g(x) = \begin{cases} 0, \ x > 0 \\ -x^2, \ x < 0. \end{cases}$$

Is your solution global in time?

4. On a bounded domain $\Omega \in \mathbb{R}^n$ consider a stationary reaction-diffusion equation

$$-\Delta u = u(1-u), \ u(x) = g(x), \ \text{on } \partial\Omega, \ 1/2 \le g(x) \le 1.$$

Prove that there is at most one (classical) solution u of this equation with the prperty $u \ge 1/2$.
Note: this problem seems to be out of place. Indeed it is not on the method of characteristics, but it is on the linearization idea we used in the proof of Theorem 7 on p.132.

5. Problem 10 from Evans on p.164