# Homework 1, Due September 5th

1. Problem 1 (Chapter 2) on p.85 of Evans's book.
2. Problem 4 (Chapter 1) on p.12-13 of Evans's book.
3. Prove the one-dimensional Harnack's inequality: Suppose $u \in C^2([-R, R])$ satisfies $u'' = 0$, and $u(x) \geq 0$. Then we have

$$\left(1 - \frac{|x|}{R}\right) u(0) \leq u(x) \leq \left(1 + \frac{|x|}{R}\right) u(0).$$

4. Suppose a positive differentiable function $E(t)$ satisfies an inequality:

$$E'(t) \leq \phi(t) E(t) \log(1 + E(t)) + \psi(t),$$

where positive functions $\phi$ and $\psi$ are uniformly bounded: for all $t \geq 0$

$$0 < \phi(t) \leq C, \; 0 < \psi(t) \leq C. \tag{1}$$

Prove that $E(t)$ does not blow up in finite time.
Hint: A Gronwall's inequality (see p.625 of book) says that if

$$E'(t) \leq \phi(t) E(t) + \psi(t),$$

with the same assumptions on $E(t)$, $\phi$ and $\psi$, then

$$E(t) \leq C e^{Ct}.$$

for some constant C[1]. In our problem we can obtain at most a "double-exponential" growth:

$$E(t) \leq C e^{C e^{Ct}}.$$

[1]As it is customary in PDE theory we will use C as a generic constant independent of any parameters of the problem. In particular this C is different from the C in equation (1)

5. Prove that the transport equation

$$\begin{cases} u_t + b \cdot \mathrm{D}u = 0, \text{ on } \mathbb{R}^n \times (0, \infty), \\ u(x, t = 0) = g(x) \in C^1(\mathbb{R}^n), \end{cases}$$

is well-posed for classical solutions.
Hint: We clearly know that the solution exists - we gave an explicit formula in class. Continuous dependence on initial conditions will also follow from the same formula and linearity of the equation (you still need to prove it, though). So, the only subtle point here is to prove uniqueness of the solution. Again, by linearity, we only need to investigate the case

$$\begin{cases} u_t + b \cdot \mathrm{D}u = 0, \text{ on } \mathbb{R}^n \times (0, \infty), \\ u(x, t = 0) = 0. \end{cases}$$

and show that $u(x, t) \equiv 0$ is the only classical[2] solution. Make a change of variables : $(x, t) \to (y, s)$, where $s = t$, $y = x - bt$, derive

$$\begin{cases} u_s = 0, \text{ on } \mathbb{R}^n \times (0, \infty), \\ u(y, s = 0) = 0, \end{cases}$$

and prove uniqueness appealing to the ODE theory.

[2]Here it means that we can assume that $u$ is continuously differentiable in $x$ and $t$.